

# Equivariant aspects of de-completing cyclic homology\*

BY ZHOUHANG MAO

## Abstract

Derived de Rham cohomology turns out to be important in  $p$ -adic geometry, following Bhatt’s discovery [Bha12] of conjugate filtration in char  $p$ , de-Hodge-completing results in [Bei12]. In [Kal18], Kaledin introduced an analogous de-completion of the periodic cyclic homology, called the polynomial periodic cyclic homology, equipped with a conjugate filtration in char  $p$ , and expected to be related to derived de Rham cohomology.

In this article, using genuine equivariant homotopy structure on Hochschild homology as in [ABG+18, BHM22], we give an equivariant description of Kaledin’s polynomial periodic cyclic homology. This leads to Morita invariance without any Noetherianness assumption as in [Kal18], and the comparison to derived de Rham cohomology becomes transparent. Moreover, this description adapts directly to “topological” analogues, which gives rise to a de-Nygaard-completion of the topological periodic cyclic homology, which admits an extension to linear categories over truncated Brown–Peterson spectra.

As an application, we establish a noncommutative crystalline–de Rham comparison, which de-completes the result in [PV19], and extends it to prime  $p = 2$ . We also compare polynomial periodic cyclic homology to topological Hochschild homology over  $\mathbb{F}_p$ , and produce a conjugate filtration in char  $p$  from our description.

## 1 Introduction

Grothendieck’s algebraic de Rham cohomology, introduced in [Gro66], turns out to be an important tool to study the cohomology of smooth schemes. However, it does not behave well beyond the smooth case. Illusie, following ideas of Quillen, introduced *derived de Rham cohomology*, along with its Hodge-completion, in [Ill72, Ch. VIII].

Hodge-completion makes derived de Rham cohomology easier to control, since the associated graded pieces are given by shifts of (derived) exterior powers of the cotangent complex. In particular, it coincides with algebraic de Rham cohomology for smooth schemes.

On the other hand, (non-Hodge-completed) derived de Rham cohomology was intractable until Bhatt’s discovery in [Bha12] of *conjugate filtration* on it in char  $p$ , whose associated graded pieces are equivalent to shifts of Frobenius twists of algebraic differential forms. He also observed the triviality of derived de Rham cohomology after rationalization. Using this new tool, he identified derived de Rham cohomology with crystalline cohomology for lci maps between  $\mathbb{Z}/p^r$ -schemes. Later on, this non-Hodge-complete version becomes useful in  $p$ -adic geometry. For example, Fontaine’s period rings  $A_{\text{cris}}$  and  $C_{\text{st}}$  are equipped with non-complete Hodge-filtration, and Bhatt applied this non-Hodge-complete version to prove some Beilinson’s conjectures in [Bei12].

Periodic cyclic homology is a noncommutative counterpart of Hodge-completed derived de Rham cohomology, defined for general DG-categories. For morphisms of  $\mathbb{Q}$ -schemes, the relation is particularly simple: periodic cyclic homology is a product of shifts of derived de Rham cohomology, as recently shown by Konrad BALS in full generality in [Bal24]. This relation was firstly discovered by Loday–Quillen for smooth morphisms of  $\mathbb{Q}$ -schemes, cf. [Lod98, §5.1.12], and studied in [TV11]. For schemes beyond char 0, it was studied in [Maj96]. Breakthroughs were made in [BMS19, Ant19], which proved that, there is a complete filtration on periodic cyclic homology, whose associated graded pieces are shifts of Hodge-completed derived de Rham cohomology.

---

\*. This article has been written using GNU  $\text{\TeX}_{\text{MACS}}$  [Hoe20].

In view of usefulness of non-Hodge-completed derived de Rham cohomology in  $p$ -adic geometry, it is natural to ask whether there is a “non-Hodge-completion” of periodic cyclic homology for DG-categories, which carries a filtration with associated graded pieces being shifts of non-Hodge-completed derived de Rham cohomology? In [Kal18], following Kontsevich’s suggestion [Kon08, 2.32], Kaledin defined *polynomial periodic cyclic homology*, equipped with a conjugate filtration in char  $p$  (when  $p=2$ , it is later constructed in [Kal17]), whose associated graded pieces are equivalent to shifts of Frobenius twisted Hochschild homology. Using this, he showed that a certain completion of polynomial periodic cyclic homology, called *co-periodic cyclic homology*, is a derived Morita invariant when the base is Noetherian. Moreover, he expected that polynomial periodic cyclic homology is closely related to derived de Rham cohomology.

As explained in [Kal18], Kaledin’s defines polynomial periodic cyclic homology and deals with it by explicit manipulations of chain complexes, which makes the arguments technical and difficult, and the homotopy-theoretic functoriality of this construction becomes opaque. The main goal of this article is to give a “more invariant treatment” as he wished in the introduction, which overcomes these difficulties.

The key to our description is the genuine equivariant homotopy structure on the usual  $\mathbb{Z}$ -linear Hochschild homology. More precisely, let  $\mathcal{C}$  be a DG-category. Then the Hochschild homology of  $\mathcal{C}$ , being a Borel  $\mathbb{T}$ -equivariant  $\mathbb{Z}$ -module spectrum, has the formula

$$\mathrm{HH}(\mathcal{C}/\mathbb{Z}) = \mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(\mathbb{Z})}^{\mathbb{L}} \mathbb{Z}.$$

This admits an obvious cyclonic (à la [BG16]) structure: the  $\mathbb{T}$ -equivariant ring  $\mathbb{Z}$  is the underlying object of the constant Tambara functor  $\underline{\mathbb{Z}}$ , and the universal property of  $\mathrm{THH}$  in [ABG+18] gives rise to a map  $\mathrm{THH}(\mathbb{Z}) \rightarrow \underline{\mathbb{Z}}$  of  $\mathbb{T}$ - $\mathbb{E}_\infty$ -rings. This gives rise to an enhancement

$$\mathrm{HH}(\mathcal{C}/\underline{\mathbb{Z}}) = \mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(\underline{\mathbb{Z}})}^{\mathbb{L}} \underline{\mathbb{Z}}$$

as a  $\underline{\mathbb{Z}}$ -module in cyclonic spectra. This was generalized to a genuine version of factorization homology in [BHM22]. However, up to our knowledge, such a genuine equivariant homotopy structure on  $\mathbb{Z}$ -linear Hochschild homology does not seem to be studied in the literature. By definition, such an enhancement is a derived Morita invariant.

**Remark 1.1.** This genuine equivariant homotopy structure has other applications. In a companion paper [Mao24b], we use a “thickening” of it to define prismatic Hochschild homology. In our paper [Mao24a], we use a similar genuine equivariant structure to streamline Kaledin’s Hochschild–Witt homology, a noncommutative counterpart of de Rham–Witt complex.

It turns out that such a structure contains enough information to recover polynomial periodic cyclic homology. Recall that, in terms of explicit chain complexes, the usual Tate construction involves a product totalization in one direction, and Kaledin’s polynomial periodic cyclic homology is taking the direct sum totalization instead, so that it has good colimit-preserving properties. Inspired by this, we introduce the following definitions:

**Definition 1.2. (Definitions 2.1 and 3.3)** *The  $(\underline{\mathbb{Z}})$ -de-completed  $\mathbb{T}$ -Tate construction  $(-)^{\theta_{\underline{\mathbb{Z}}} \mathbb{T}}$  is the filtered-colimit-preserving approximation of the composite functor*

$$\mathrm{Mod}_{\underline{\mathbb{Z}}}(\mathrm{Sp}^{g < \mathbb{T}}) \longrightarrow \mathrm{Sp}^{B\mathbb{T}} \xrightarrow{(-)^{t\mathbb{T}}} \mathrm{Sp}.$$

*The polynomial periodic cyclic homology  $\mathrm{HP}^{\mathrm{poly}}(\mathcal{C}/\underline{\mathbb{Z}})$  of a DG-category  $\mathcal{C}$  is defined to be  $\mathrm{HH}(\mathcal{C}/\underline{\mathbb{Z}})^{\theta_{\underline{\mathbb{Z}}} \mathbb{T}}$ , applying the de-completed  $\mathbb{T}$ -Tate construction to Hochschild homology  $\mathrm{HH}(\mathcal{C}/\underline{\mathbb{Z}})$ .*

The same construction works for any  $t$ -bounded (i.e. bounded with respect to the  $t$ -structure) animated ring as base in place of  $\mathbb{Z}$ , as done in the main text.

**Remark 1.3.** The de-completed Tate construction depends on the choice of base. However, in some cases, it does not quite depend on that. We will prove relevant results in Section 4.

From this description, it is immediate that polynomial periodic cyclic homology is rationally zero (Remark 2.9), since the functor  $\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}^{g<\mathbb{T}}) \rightarrow \mathrm{Sp}^{B\mathbb{T}}$  becomes an equivalence after rationalization. With slightly more efforts, we show that

**Proposition 1.4. (Proposition 4.4)** *Let  $\mathcal{C}$  be a smooth and bounded DG-category<sup>1.1</sup>. Then the assembly map  $\mathrm{HP}^{\mathrm{poly}}(\mathcal{C}/\mathbb{Z}) \rightarrow \mathrm{HP}(\mathcal{C}/\mathbb{Z})$  is an equivalence after profinite completion.*

Note that, for every quasicompact quasiseparated scheme  $X$ , its derived category  $D(X)$  is bounded, by [BvdB03, Cor 3.1.8]. When  $X$  is in addition smooth, then its derived category  $D(X)$  is also smooth. We refer to [Orl16] for general discussions. This proposition, along with colimit-preservation of de-completed Tate construction, implies that, on animated rings, the polynomial periodic cyclic homology is left Kan extended from polynomial rings, and thus it coincides with various adhoc constructions in the literature<sup>1.2</sup>, such as in [BMS19, AMMN22]. Consequently, we address Kaledin’s expectation in the following.

**Proposition 1.5. (Construction 4.8)** *Let  $R$  be a commutative ring. Then there exists a functorial filtration  $\mathrm{Fil}_{\mathrm{HKR}}^*$  on the profinite completion  $\mathrm{HP}^{\mathrm{poly}}(R/\mathbb{Z})^\wedge$  of polynomial periodic cyclic homology with associated graded pieces equivalent to shifts of derived de Rham cohomology  $\mathrm{dR}_{R/\mathbb{Z}}$  after profinite completion.*

Our description also suggests a “topological” analogue.

**Definition 1.6. (Definition 4.9)** *Let  $S$  be a perfectoid ring, and  $\mathcal{C}$  a DG-category over  $S$ . Then topological polynomial periodic cyclic homology  $\mathrm{TP}^{\mathrm{poly}/S}(\mathcal{C})$  is defined to be  $\mathrm{THH}(\mathcal{C})^{\theta_{\mathrm{THH}(S)}\mathbb{T}}$ , defined by applying  $\mathrm{THH}(S)$ -de-completed  $\mathbb{T}$ -Tate construction to topological Hochschild homology  $\mathrm{THH}(\mathcal{C})$ .*

Note that this is an arena where explicit chain complex manipulations cannot arrive. Previous results for polynomial periodic cyclic homology adapts to its topological analogue as well:

**Proposition 1.7. (Corollary 4.12)** *Let  $S$  be a perfectoid ring, and  $R$  a  $p$ -completely smooth and bounded DG-category over  $S$ . Then the assembly map  $\mathrm{TP}^{\mathrm{poly}/S}(R) \rightarrow \mathrm{TP}(R)$  is an equivalence after  $(p, \ker(\theta))$ -completion, where  $\theta: A_{\mathrm{inf}}(S) \twoheadrightarrow S$  is Fontaine’s map.*

**Proposition 1.8. (Construction 4.13)** *Let  $S$  be a perfectoid ring, and  $R$  a commutative  $S$ -algebra. Then there exists a functorial filtration  $\mathrm{Fil}_M^*$  on the  $(p, \ker(\theta))$ -completed topological polynomial periodic cyclic homology  $\mathrm{TP}^{\mathrm{poly}/S}(R)_{(p, \ker(\theta))}^\wedge$  with associated graded pieces equivalent to shifts of Frobenius twisted prismatic cohomology  $\varphi_A^* \Delta_{R/A}$  after  $(p, \ker(\theta))$ -completion, where  $A := A_{\mathrm{inf}}(S)$ .*

In a forthcoming work of Devalapurkar–Hahn–Raksit–Yuan, they produce de-completions of  $\mathrm{TC}^-$  and  $\mathrm{TP}$  which carry filtrations with associated graded pieces given by *absolute prismatic cohomology* and its Nygaard filtered pieces. Our techniques adapt to this situation as well, giving rise to *de-completed Borel completion*  $(-)^{\eta_{\mathrm{THH}(\mathbb{Z})}}: \mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})}(\mathrm{Sp}^{g<\mathbb{T}}) \rightarrow \mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})^h}(\mathrm{Sp}^{g<\mathbb{T}})$ .

**Theorem 1.9. (Corollary 5.10)** *The functor*

$$\mathrm{THH}_\Delta: \mathrm{CAlg}_{\mathbb{Z}}^{\mathrm{an}} \rightarrow \mathrm{RMod}_{\mathrm{THH}(\mathbb{Z})^h}(\mathrm{Sp}^{g<\mathbb{T}})_p^\wedge$$

*in Devalapurkar–Hahn–Raksit–Yuan extends to a localizing invariant over  $\mathbb{Z}$ , which is equivalent to  $(\mathrm{THH}(-)^{\eta_{\mathrm{THH}(\mathbb{Z})}})_p^\wedge$ . We can replace  $\mathbb{Z}$  by truncated Brown–Peterson spectra  $\mathrm{BP}\langle n \rangle$  for  $n \in \mathbb{N}$  as well (Remark 5.32).*

1.1. Smooth DG-categories are necessarily modules categories over smooth  $\mathbb{E}_1$ -rings, by [Lur18, Prop 11.3.2.4]. A DG-category is *bounded* if, for every pair  $(x, y)$  of compact objects, the mapping  $\mathbb{Z}$ -module spectrum  $\mathrm{Hom}(x, y) \in D(\mathbb{Z})$  has bounded Tor-amplitude (or equivalently,  $t$ -bounded, since  $\mathbb{Z}$  is of finite flat dimension).

1.2. Also compare with [Man24, §1].

Thus this extended  $\mathrm{THH}_{\Delta}$  is indeed a noncommutative counterpart of (Nygaard-filtered) absolute prismatic cohomology. Actually, we can recover  $\mathrm{TP}^{\mathrm{poly}/S}$  from this new construction:

**Theorem 1.10. (Example 5.29 and Corollary 5.31)** *Let  $S$  be a perfectoid ring, and  $\mathcal{C}$  a DG-category over  $S$ . Then we have equivalences*

$$\begin{aligned} ((\mathrm{THH}(\mathcal{C})^{\eta_{\mathrm{THH}(\mathbb{Z})}})^{C_{p^\infty}})_{(p,v_1)}^\wedge &\simeq \mathrm{TC}^{-,\mathrm{poly}/S}(\mathcal{C})_{(p,\xi)}^\wedge, \\ (((\mathrm{THH}(\mathcal{C})^{\eta_{\mathrm{THH}(\mathbb{Z})}})^{\Phi C_p})^{C_{p^\infty}/C_p})_{(p,v_1)}^\wedge &\simeq \mathrm{TP}^{\mathrm{poly}/S}(\mathcal{C})_{(p,\xi)}^\wedge. \end{aligned}$$

It seems slightly surprising that cyclotomic  $\mathrm{THH}$  (with  $\mathrm{THH}(\mathbb{Z})$ -module structure) is already enough to de-Nygaard-complete topological periodic cyclic homology, but this phenomenon is demystified by Efimov’s rigidity of localizing motives (Remark 4.17).

As for noncommutative geometry on its own, we first give a noncommutative crystalline–de Rham comparison. Let  $R$  be an animated ring. Then the crystalline–de Rham comparison theorem tells us that the crystalline cohomology of  $R \otimes_{\mathbb{Z}} \mathbb{F}_p$  over  $\mathbb{Z}_p$  is equivalent to the  $p$ -completion of the (non-Hodge-completed) de Rham of  $R$  over  $\mathbb{Z}$ . In [PV19], when  $p$  is an odd prime, for DG-categories  $\mathcal{C}$ , they give a comparison between  $\mathrm{TP}(\mathcal{C} \otimes_{\mathbb{Z}} \mathbb{F}_p)$  with  $\mathrm{HP}(\mathcal{C}/\mathbb{Z})$ , which corresponds to such a comparison under certain completions. This result is improved by [DR25, Thm 0.5.1] as well for odd primes  $p$ . Equipped with previous constructions, adapting main ideas in [PV19] along with an observation by A. RAKSIT, we prove a comparison of de-completed constructions which corresponds to the crystalline–de Rham comparison for all primes:

**Theorem 1.11. (Corollary 6.28)** *Let  $\mathcal{C}$  be a DG-category. Then there exists a lax symmetric monoidal (in  $\mathcal{C}$ ) equivalence*

$$\mathrm{TP}^{\mathrm{poly}/\mathbb{F}_p}(\mathcal{C} \otimes_{\mathbb{Z}} \mathbb{F}_p)_p^\wedge \simeq \mathrm{HP}^{\mathrm{poly}}(\mathcal{C}/\mathbb{Z})_p^\wedge.$$

We then give two proofs for the following comparison, due to Kaledin in [Kal20, Cor 11.15], but our proof is much simpler.

**Proposition 1.12. (Corollary 7.10)** *Let  $\mathcal{C}$  be a DG-category over  $\mathbb{F}_p$ . Then the polynomial periodic cyclic homology  $\mathrm{HP}^{\mathrm{poly}}(\mathcal{C}/\mathbb{F}_p)$  is equivalent to  $\mathrm{THH}(\mathcal{C})[\sigma^{-1}]$  as  $\mathbb{Z}^{t\mathbb{T}}$ -module spectra.*

We also produce a conjugate filtration on polynomial periodic cyclic homology in char  $p$  in Section 8, and prove that

**Proposition 1.13. (Corollary 8.21)** *Let  $k$  be a commutative  $\mathbb{F}_p$ -algebra, and  $\mathcal{C}$  a DG-category over  $k$ . Then the conjugate filtration on  $\mathrm{HP}^{\mathrm{poly}}(\mathcal{C}/k)$  is complete in the following two cases:*

1.  $\mathcal{C} = D(R)$  for some  $(-1)$ -connective  $\mathbb{E}_1$ - $k$ -algebra  $R$  (which includes all associative  $k$ -algebras  $R$ );
2.  $\mathcal{C} = D(X)$  for a quasicompact quasiseparated  $k$ -scheme  $X$ .

**Remark 1.14.** In comparison Efimov’s refined negative cyclic homology and its continuation in Scholze’s ongoing work<sup>1.3</sup> on refined  $\mathrm{TC}^-$ , as mentioned above, in Remark 4.17, we explain that topological Hochschild homology as a cyclotomic spectrum already sees “all”  $p$ -adic formal information. However, their versions capture rigid analytic information. For example, for smooth  $\mathbb{F}_p$ -schemes, their versions see rigid cohomology. It might be worth understanding whether equivariant homotopy theory could say something for their versions as well.

**Notation 1.15.** Let  $G$  be a finite group. We denote by  $\mathrm{Sp}^{gG}$  the symmetric monoidal  $\infty$ -category of genuine  $G$ -spectra, by  $\mathrm{Mack}_G^{\mathrm{coh}}(k)$  the abelian category of  $k$ -linear cohomological  $G$ -Mackey functors (we omit  $k$  when  $k = \mathbb{Z}$ ), and by  $\mathrm{Sp}^{g^{\leq \mathbb{T}}}$  (resp.  $\mathrm{Sp}^{g^p \mathbb{T}}$ ) the symmetric monoidal  $\infty$ -category of cyclonic (resp.  $p$ -cyclonic) spectra as in [BG16].

<sup>1.3</sup>. An abstract, along with recordings, of the talk can be found at <https://www.mpim-bonn.mpg.de/node/13359>.

**Acknowledgments.** We would like to thank Lukas BRANTNER, Christian CARRICK, Kaif HILMAN, Dmitry KALEDIN, Kirill MAGIDSON, Alexander PETROV, Arpon RAKSIT, Maxime RAMZI, Georg TAMME, and Dmitry VAINTROB. We also thank Robert BURKLUND and Jeremy HAHN. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 864145).

## 2 A de-completion of Tate construction

Let  $k$  be a commutative ring, and  $G$  a finite group or  $\mathbb{T}$ . Recall that the  $G$ -Tate construction  $(-)^{tG}: D(k)^{BG} \rightarrow D(k)$  does not preserve filtered colimits. In this section, we try to “de-complete” it when the input is further equipped with a genuine equivariant structure. When  $G = C_p$ , we will show that it can be expressed in terms of the geometric fixed points. We will also establish a de-completed version of the Tate orbit lemma.

**Definition 2.1.** Let  $G$  be a finite group (resp.  $\mathbb{T}$ ), and  $A$  an  $\mathbb{E}_1$ -algebra in the symmetric monoidal  $\infty$ -category  $\mathrm{Sp}^{gG}$  (resp.  $\mathrm{Sp}^{g<\mathbb{T}}$ ) of  $G$ -spectra (resp. cyclonic spectra). Then

- The  $(A)$ -de-completed homotopy  $G$ -fixed points  $(-)^{\eta_A G}: \mathrm{RMod}_A(\mathrm{Sp}^{gG}) \rightarrow \mathrm{Sp}$  (resp.  $\mathrm{RMod}_A(\mathrm{Sp}^{g<\mathbb{T}}) \rightarrow \mathrm{Sp}$ ) is the filtered-colimit-preserving approximation of the homotopy  $G$ -fixed points  $(-)^{hG}: \mathrm{RMod}_A(\mathrm{Sp}^{gG}) \rightarrow \mathrm{Sp}$  (resp.  $\mathrm{RMod}_A(\mathrm{Sp}^{g<\mathbb{T}}) \rightarrow \mathrm{Sp}$ ), equipped with an assembly map  $(-)^{\eta_A G} \rightarrow (-)^{hG}$ . We omit “ $A$ -” when the context is clear.
- The  $(A)$ -de-completed  $G$ -Tate construction  $(-)^{\theta_A G}: \mathrm{RMod}_A(\mathrm{Sp}^{gG}) \rightarrow \mathrm{Sp}$  (resp.  $\mathrm{RMod}_A(\mathrm{Sp}^{g<\mathbb{T}}) \rightarrow \mathrm{Sp}$ ) is the filtered-colimit-preserving approximation of the  $G$ -Tate construction  $(-)^{tG}: \mathrm{RMod}_A(\mathrm{Sp}^{gG}) \rightarrow \mathrm{Sp}$  (resp.  $\mathrm{RMod}_A(\mathrm{Sp}^{g<\mathbb{T}}) \rightarrow \mathrm{Sp}$ ), equipped with an assembly map  $(-)^{\theta_A G} \rightarrow (-)^{tG}$  which canonically fits into a commutative diagram

$$\begin{array}{ccc} (-)^{\eta_A G} & \longrightarrow & (-)^{hG} \\ \downarrow & & \downarrow \\ (-)^{\theta_A G} & \longrightarrow & (-)^{tG} \end{array} . \quad (2.1)$$

**Remark 2.2.** Let  $G$  be a finite cyclic group (resp.  $\mathbb{T}$ ), and  $A$  a commutative algebra in  $\mathcal{E} := \mathrm{Sp}^{gG}$  (resp.  $\mathcal{E} := \mathrm{Sp}^{g<\mathbb{T}}$ ). Then the diagram (2.1) is Cartesian in the  $\infty$ -category  $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{E}, \mathrm{Sp})$ : the fiber of the canonical natural transformation  $(-)^{hG} \rightarrow (-)^{tG}$  is  $(-)_{hG} \in \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{E}, \mathrm{Sp})$ , which preserves filtered colimits as well, and consequently, the diagram (2.1) induces an equivalence on fibers of vertical arrows.

**Remark 2.3.** Let  $G$  be a finite group (resp.  $\mathbb{T}$ ), and  $A \rightarrow B$  a map of  $\mathbb{E}_1$ -algebras in the symmetric monoidal  $\infty$ -category  $\mathcal{E} := \mathrm{Sp}^{gG}$  (resp.  $\mathcal{E} := \mathrm{Sp}^{g<\mathbb{T}}$ ) of  $G$ -spectra (resp. cyclonic spectra). Then the assembly maps on compact objects of  $\mathrm{RMod}_B(\mathcal{E})$  induces “relative” assembly maps  $(-)^{\eta_A G} \rightarrow (-)^{\eta_B G}$  and  $(-)^{\theta_A G} \rightarrow (-)^{\theta_B G}$ , which fits into a commutative diagram

$$\begin{array}{ccc} (-)^{\eta_A G} & \longrightarrow & (-)^{\eta_B G} \\ \downarrow & & \downarrow \\ (-)^{\theta_A G} & \longrightarrow & (-)^{\theta_B G} \end{array}$$

which is Cartesian by Remark 2.2.

**Remark 2.4.** Let  $G$  be a finite group (resp.  $\mathbb{T}$ ), and  $A$  an  $\mathbb{E}_1$ -algebra in the symmetric monoidal  $\infty$ -category  $\mathcal{E} := \mathrm{Sp}^{gG}$  (resp.  $\mathcal{E} := \mathrm{Sp}^{g<\mathbb{T}}$ ) of  $G$ -spectra (resp. cyclonic spectra). Let  $A \rightarrow A^h$  denote the Borel completion of  $A$ . Then it follows immediately from the definitions that the de-completed homotopy  $G$ -fixed points  $(-)^{\eta_A G}$  factors as

$$\mathrm{RMod}_A(\mathcal{E}) \xrightarrow{(-) \otimes_A^{\mathbb{L}} A^h} \mathrm{RMod}_{A^h}(\mathcal{E}) \xrightarrow{\eta_{A^h} G} \mathrm{Sp}$$

and similarly for the de-completed  $G$ -Tate construction. Roughly speaking, it does not hurt to replace all genuine equivariant bases by their Borel completions. However, sometimes it seems to be convenient to consider genuine equivariant bases.

**Remark 2.5.** Let  $G$  be a finite group, and  $A$  an  $\mathbb{E}_\infty$ -algebra in  $\mathrm{Sp}^{gG}$ . Then the lax symmetric monoidal structure on the homotopy fixed points  $(-)^{hG}$  (resp. the Tate construction  $(-)^{tG}$ ) gives rise<sup>2.1</sup> to a lax symmetric monoidal structure on the de-completed homotopy fixed points  $(-)^{\eta_A G}$  (resp. the de-completed Tate construction  $(-)^{\theta_A G}$ ). The assembly maps are equipped with a lax symmetric monoidal structure as well. In particular, the objects  $M^{\eta_A G}$  (resp.  $M^{\theta_A G}$ ) carries a canonical  $A^{hG}$ -(resp.  $A^{tG}$ -)module structure, which is functorial in  $M \in \mathrm{Mod}_A(\mathrm{Sp}^{gG})$ .

**Remark 2.6.** In desirable situations, the  $A$ -de-completed homotopy  $G$ -fixed points (resp.  $G$ -Tate construction) does not quite depend on  $A$ . We discuss some independences of this form in Section 4.

**Remark 2.7.** Let  $G$  be a finite group, and  $A$  an  $\mathbb{E}_\infty$ -algebra in  $\mathrm{Sp}^{gG}$ . Recall that the symmetric monoidal  $\infty$ -category  $\mathrm{Sp}^{gG}$  is rigid, thus the forgetful functor  $\mathrm{Mod}_A(\mathrm{Sp}^{gG}) \rightarrow \mathrm{Mod}_A(\mathrm{Sp}^{BG})$  factors through the rigidification  $(\mathrm{Mod}_A(\mathrm{Sp}^{BG}))^{\mathrm{rig}}$  of the target, and the  $A$ -de-completed  $G$ -Tate construction  $(-)^{\theta_A G}$  coincides with the composite

$$\mathrm{Mod}_A(\mathrm{Sp}^{gG}) \longrightarrow (\mathrm{Mod}_A(\mathrm{Sp}^{BG}))^{\mathrm{rig}} \xrightarrow{(-)^{tG, \mathrm{rig}}} \mathrm{Sp},$$

which can be checked by restricting to compact objects of  $\mathrm{Mod}_A(\mathrm{Sp}^{gG})$ . The same holds for de-completed homotopy  $G$ -fixed points.

**Remark 2.8.** Let  $A$  be an  $\mathbb{E}_\infty$ -algebra in  $D(\mathbb{Z}) \otimes \mathrm{Sp}^{g < \mathbb{T}}$ . Recall that, for every  $A$ -module  $M$  in  $\mathrm{Sp}^{g < \mathbb{T}}$  and every positive integer  $n \in \mathbb{N}_{>0}$ , the canonical maps

$$M^{t\mathbb{T}}/n \longleftarrow M^{t\mathbb{T}} \otimes_{A^{t\mathbb{T}}} A^{tC_n} \longrightarrow M^{tC_n}$$

are equivalences, [NS18, Lem IV.4.12]. It follows that, the canonical maps

$$M^{\theta_A \mathbb{T}}/n \longleftarrow M^{\theta_A \mathbb{T}} \otimes_{A^{t\mathbb{T}}} A^{tC_n} \longrightarrow M^{\theta_A C_n}$$

are equivalences as well. Consequently, up to profinite completion, the  $A$ -de-completed  $\mathbb{T}$ -Tate construction  $M^{\theta_A \mathbb{T}}$  can be recovered from de-completed  $C_n$ -Tate constructions, where  $n$  runs through all positive integers. This gives rise to a lax symmetric monoidal structure on the profinitely completed de-completed  $\mathbb{T}$ -Tate construction  $(M^{\theta_A \mathbb{T}})^\wedge$ .

**Remark 2.9.** Let  $k$  be a commutative ring. Then the constant Green functor  $\underline{k}$  can be viewed as an object of  $D(\mathbb{Z}) \otimes \mathrm{Sp}^{g < \mathbb{T}}$ . Then for every  $n \in \mathbb{N}_{>0}$ , we have

$$(\underline{k} \otimes \Sigma_{\mathbb{T}}^\infty [\mathbb{T}/C_n]_+)^{h\mathbb{T}} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq 0 \simeq (\underline{k} \otimes \Sigma_{\mathbb{T}}^\infty [\mathbb{T}/C_n]_+)^{t\mathbb{T}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Consequently, the rationalized de-completed  $\mathbb{T}$ -Tate construction  $(-)^{\theta_{\underline{k}} \mathbb{T}}$  vanishes. Combining with Remark 2.8, we see that the de-completed  $\mathbb{T}$ -Tate construction  $(-)^{\theta_{\underline{k}} \mathbb{T}}$  can be completely recovered from de-completed  $C_n$ -Tate constructions, where  $n$  runs through all positive integers.

The Tate orbit lemma admits a de-completion. We first observe that, since  $C_p$  is a simple group<sup>2.2</sup>, the de-complete  $C_p$ -Tate construction has a fairly simple formula:

**Lemma 2.10.** *Let  $A$  be an  $\mathbb{E}_1$ -algebra in the symmetric monoidal category  $\mathrm{Sp}^{gC_p}$  of  $C_p$ -spectra. Then the  $A$ -de-completed  $C_p$ -Tate construction  $(-)^{\theta_A C_p}$  coincides with the composite functor*

$$\mathrm{RMod}_A(\mathrm{Sp}^{gC_p}) \xrightarrow{(-)^{\Phi_{C_p}}} D(A^{\Phi_{C_p}}) \longrightarrow D(A^{tC_p})$$

2.1. Here the same argument does not work for  $G = \mathbb{T}$ .

2.2. In this article, we only consider cyclic group actions, but the argument works for any simple group.



where the second functor is the base change along the map  $A^{\Phi_{C_p}} \rightarrow A^{t_{C_p}}$  of  $\mathbb{E}_1$ -rings. Moreover, when  $A$  is an  $\mathbb{E}_\infty$ -algebra in  $\mathrm{Sp}^{g_{C_p}}$ , this identification is (lax) symmetric monoidal.

**Proof.** It suffices to restrict to compact objects of  $\mathrm{RMod}_A(\mathrm{Sp}^{g_{C_p}})$ . For compact objects, there are many ways to see this. For example, the functor  $(-)^{t_{C_p}}$  vanishes on induced  $C_p$ -spectra, thus it canonically factors through  $D(A^{\Phi_{C_p}})$  in  $\mathrm{Pr}^L$  (here we use the fact that  $C_p$  is a simple group), cf. [AMR21, §5], and then to see that the result functor  $D(A^{\Phi_{C_p}}) \rightarrow D(A^{t_{C_p}})$  in  $\mathrm{Pr}^L$  coincides with the base change, it suffices to check on the generator, which is straightforward. The symmetric monoidal structure follows from a similar argument.  $\square$

**Corollary 2.11.** *Let  $A$  be an  $\mathbb{E}_1$ -algebra in the symmetric monoidal category  $\mathrm{Sp}^{g_{C_p}}$  of  $C_p$ -spectra. Then the  $A$ -de-completed  $C_p$ -Tate construction  $(-)^{\theta_A C_p} : \mathrm{RMod}_A(\mathrm{Sp}^{g_{C_p}}) \rightarrow D(A^{t_{C_p}})$  is strongly continuous<sup>2.3</sup>.*

**Corollary 2.12.** *Let  $A$  be an  $\mathbb{E}_\infty$ -algebra in the symmetric monoidal category  $\mathrm{Sp}^{g_{C_p}}$  of  $C_p$ -spectra. Then the  $A$ -de-completed  $C_p$ -Tate construction  $(-)^{\theta_A C_p}$  is symmetric monoidal.*

**Lemma 2.13.** *Let  $A$  be a bounded below  $\mathbb{E}_1$ -algebra in cyclonic spectra. Then the  $p$ -completed  $A$ -de-completed  $\mathbb{T}$ -Tate construction<sup>2.4</sup>*

$$((-)^{\theta_A \mathbb{T}})_p^\wedge : \mathrm{RMod}_A(\mathrm{Sp}^{g^{< \mathbb{T}}}) \longrightarrow D(A^{t \mathbb{T}})_p^\wedge,$$

factors as<sup>2.5</sup>

$$\mathrm{RMod}_A(\mathrm{Sp}^{g^{< \mathbb{T}}}) \xrightarrow{(-)^{\theta_A C_p}} \mathrm{RMod}_{A^{t_{C_p}}}(\mathrm{Sp}^{g_p(\mathbb{T}/C_p)}) \xrightarrow{(-)^{\eta_{A^{t_{C_p}}}(\mathbb{T}/C_p)}} D(A^{t \mathbb{T}})_p^\wedge$$

in  $\mathrm{Pr}_{\mathrm{St}}^L$ .

**Proof.** Since every functor preserves filtered colimits, it suffices to check on compact objects in  $\mathrm{RMod}_A(\mathrm{Sp}^{g^{< \mathbb{T}}})$ , and this follows from Corollary 2.11 and the Tate orbit lemma (since  $A$  is bounded below, so is  $A \otimes \Sigma_{\mathbb{T}}^\infty [\mathbb{T}/C_m]_+$  for every  $m \in \mathbb{N}_{>0}$ ).  $\square$

The same argument works for finite cyclic groups (for which we can further keep track of the lax symmetric monoidal structure on the de-completed Tate construction).

**Lemma 2.14.** *Let  $r \in \mathbb{N}_{>0}$ , and  $A$  a bounded below  $\mathbb{E}_1$ -(resp.  $\mathbb{E}_\infty$ -)algebra in  $C_{p^r}$ -spectra. Then the  $A$ -de-completed  $C_{p^r}$ -Tate construction*

$$(-)^{\theta_A \mathbb{T}} : \mathrm{Mod}_A(\mathrm{Sp}^{C_{p^r}}) \longrightarrow D(A^{t_{C_{p^r}}}),$$

as a presentable (resp. lax symmetric monoidal) functor, factors as

$$\mathrm{Mod}_A(\mathrm{Sp}^{g_{C_{p^r}}}) \xrightarrow{(-)^{\theta_A C_p}} \mathrm{Mod}_{A^{t_{C_p}}}(\mathrm{Sp}^{g_p(C_{p^r}/C_p)}) \xrightarrow{(-)^{\eta_{A^{t_{C_p}}}(C_{p^r}/C_p)}} D(A^{t_{C_{p^r}}}).$$

**Remark 2.15.** Let  $k$  be a commutative ring. By Remarks 2.8 and 2.9 and Lemma 2.14, we can also keep track of the lax symmetric monoidal structure on the  $p$ -completed de-completed  $\mathbb{T}$ -Tate construction  $((-)^{\theta_k \mathbb{T}})_p^\wedge$ , showing that, as an exact lax symmetric monoidal functor, it factors through the composite exact symmetric monoidal functor

$$\mathrm{Mod}_k(\mathrm{Sp}^{g^{< \mathbb{T}}}) \xrightarrow{(-)^{\Phi_{C_p}}} \mathrm{Mod}_{k^{\Phi_{C_p}}}(\mathrm{Sp}^{g^{< (\mathbb{T}/C_p)})} \longrightarrow \mathrm{Mod}_{k^{t_{C_p}}}(\mathrm{Sp}^{g^{< (\mathbb{T}/C_p)}),$$

where the remaining functor  $\mathrm{Mod}_{k^{t_{C_p}}}(\mathrm{Sp}^{g^{< (\mathbb{T}/C_p)})} \rightarrow D(A)$  is the limit of de-completed homotopy  $C_{p^r}$ -fixed points along  $r \in \mathbb{N}$  (which at least a priori does not necessarily preserve filtered colimits).

<sup>2.3</sup>. A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  in  $\mathrm{Pr}_{\mathrm{St}}^L$  is *strongly continuous* if its right adjoint  $F^R : \mathcal{D} \rightarrow \mathcal{C}$  preserves filtered colimits. When  $\mathcal{C}$  is compactly generated, it is equivalent to  $F$  preserving compact objects.

<sup>2.4</sup>. Note that the forgetful functor  $\mathrm{Sp}^{g^{< \mathbb{T}}} \rightarrow \mathrm{Sp}^{g_p \mathbb{T}}$  induces an equivalence on  $p$ -complete objects, thus we could work  $p$ -typically throughout.

<sup>2.5</sup>. Thanks to Remark 2.4, it does not matter what cyclonic structure on  $A^{t_{C_p}}$  that we put.

### 3 Polynomial periodic and negative cyclic homology

We briefly review Kaledin's *polynomial periodic cyclic homology* of cyclic objects, and then describe it in terms of the de-completed  $\mathbb{T}$ -Tate construction (Definition 3.3), informally explaining why it coincides with Kaledin's original construction.

Let  $k$  be a commutative ring, and  $X_\bullet : \Lambda^{\text{op}} \rightarrow \text{Mod}_k$  a cyclic objects in  $k$ -modules. Recall that, for every  $[n]_\Lambda \in \Lambda^{\text{op}}$ , the  $k$ -module  $X_n := X_\bullet([n]_\Lambda)$  carries an  $k$ -linear  $C_n$  action, which gives rise to a 2-periodic complex

$$\begin{array}{ccccccc} \cdots & \xleftarrow{1-\sigma_n} & X_n & \xleftarrow{N_n} & X_n & \xleftarrow{1-\sigma_n} & X_n & \xleftarrow{N_n} & \cdots \\ \text{weight} & & -1 & & 0 & & 1 & & \end{array}$$

of  $k$ -modules, where  $\sigma_n : X_n \rightarrow X_n$  is the generator of  $C_n$ , and  $N_n := 1 + \sigma_n + \cdots + \sigma_n^{n-1}$  is the  $C_n$ -norm. This complex represents the shifted Tate construction  $X_n^{tC_n}[-1]$ . These complexes compile into a double complex

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \xleftarrow{1-\sigma} & X_1 & \xleftarrow{N} & X_1 & \xleftarrow{1-\sigma} & X_1 & \xleftarrow{N} & \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \xleftarrow{1-\sigma} & X_0 & \xleftarrow{N} & X_0 & \xleftarrow{1-\sigma} & X_0 & \xleftarrow{N} & \cdots \end{array}$$

where we surpress the subscripts of  $N$  and  $\sigma$ , and vertical differentials are appropriately 2-periodically given. Then

- The periodic cyclic homology  $\text{HP}(X_\bullet/k)$  of the cyclic object  $X_\bullet$ , is the object in the derived category  $D(k)$  of  $k$ -modules represented by the product totalization of this double complex.
- The polynomial periodic cyclic homology  $\text{HP}^{\text{poly}}(X_\bullet/k)$  of the cyclic object  $X_\bullet$  is the object in the derived category  $D(k)$  of  $k$ -modules represented by the direct sum totalization of this double complex.

We now give an alternative, more conceptual description of this double complex and the polynomial periodic cyclic homology. Recall that, the geometric realization  $|X_\bullet|_\Lambda \in D(k)^{B\mathbb{T}}$  can be rewritten as a geometric realization

$$\text{colim}_{[n] \in \Delta^{\text{op}}} \text{Ind}_{C_n}^{\mathbb{T}}(X_n) \in D(k)^{B\mathbb{T}}.$$

If we apply the  $\mathbb{T}$ -Tate construction to it, and *incorrectly* interchange  $(-)^{t\mathbb{T}}$  with  $\text{colim}_{[n]}$ , we get

$$\text{colim}_{[n] \in \Delta^{\text{op}}} (\text{Ind}_{C_n}^{\mathbb{T}}(X_n))^{t\mathbb{T}} \in D(k)$$

where  $(\text{Ind}_{C_n}^{\mathbb{T}}(X_n))^{t\mathbb{T}} \simeq X_n^{tC_n}[-1]$  [HN20, Prop 3], thus we see that this computes the polynomial periodic cyclic homology. Note that every  $G$ -module  $M$  gives rise to a cohomological  $G$ -Mackey functor  $\underline{M} : [G/H] \mapsto M^H$ , we can endow  $C_n$ -module  $X_n$  a cohomological  $C_n$ -Mackey functor structure, and thus realize the geometric realization  $|X_\bullet|_\Lambda$  as an object of  $\text{Mod}_k(\text{Sp}^{\mathbb{T}})$ , and apply the de-completed  $\mathbb{T}$ -Tate construction  $(-)^{\theta_k \mathbb{T}}$  to it, obtaining polynomial periodic cyclic homology.

**Remark 3.1.** This procedure can be made more rigorous by considering cohomological Mackey functors over Connes' cyclic category  $\Lambda^{\text{op}}$ . Since we do not depend on Kaledin's original construction, we skip such a development. However, a toy version of this is explained in Appendix A.

**Question 1.** This comparison does not compare the lax symmetric monoidal structure of Kaledin's polynomial periodic cyclic homology and ours. How do we compare the ring structure on the two?

Now we give our formal definitions, and explain why it corresponds to the construction above for associative flat  $k$ -algebras.



**Construction 3.2.** Let  $k$  be a  $t$ -bounded<sup>3.1</sup> animated ring. The universal property of  $\mathrm{THH}$  as in [ABG+18] gives rise to a map  $\mathrm{THH}(k) \rightarrow \underline{k}$  of  $\mathbb{T}$ - $\mathbb{E}_\infty$ -rings. Thus for every  $\mathrm{THH}(k)$ -module  $M$  in cyclonic spectra, we get an object  $M \otimes_{\mathrm{THH}(k)} \underline{k} \in \mathrm{Mod}_{\underline{k}}(\mathrm{Sp}^{g^<\mathbb{T}})$ . In particular, let  $\mathcal{C}$  be a dualizable presentable stable  $k$ -linear  $\infty$ -category, we have a canonical genuine equivariant enhancement of the  $k$ -linear Hochschild homology of  $\mathcal{C}$ , denoted by  $\mathrm{HH}(\mathcal{C}/\underline{k}) \in \mathrm{Mod}_{\underline{k}}(\mathrm{Sp}^{g^<\mathbb{T}})$ .

**Definition 3.3.** Let  $k$  be a  $t$ -bounded animated ring, and  $\mathcal{C}$  a dualizable presentable stable  $k$ -linear  $\infty$ -category. Then the polynomial periodic cyclic homology  $\mathrm{HP}^{\mathrm{poly}}(\mathcal{C}/k)$  (resp. the polynomial negative cyclic homology  $\mathrm{HC}^{-,\mathrm{poly}}(\mathcal{C}/k)$ ) is defined to be the  $\underline{k}$ -de-completed  $\mathbb{T}$ -Tate construction  $\mathrm{HH}(\mathcal{C}/\underline{k})^{\theta_{\underline{k}}\mathbb{T}}$  (resp. the  $\underline{k}$ -de-completed homotopy  $\mathbb{T}$ -fixed points  $\mathrm{HH}(\mathcal{C}/\underline{k})^{\eta_{\underline{k}}\mathbb{T}}$ ).

Let  $k$  be a commutative ring, and  $R$  an associative flat  $k$ -algebra. Recall that, as in [ABG+18, §6], the Hochschild homology  $\mathrm{HH}(R/\underline{k})$  relative to the constant Tambara functor  $\underline{k}$  is informally given by the geometric realization of relative norms  $[n] \mapsto R^{\otimes_{\underline{k}}^{\mathbb{T}} C_n}$ . In [Mao24a], we formally identify these norms with the naive tensor powers equipped with the obvious cyclic action. Thus the previous discussion basically identifies the polynomial periodic cyclic homology  $\mathrm{HP}^{\mathrm{poly}}(R/k)$  with Kaledin’s original one.

## 4 de Rham and prismatic comparison

In this section, we will first establish a comparison result (Proposition 4.4), which says that, for smooth commutative algebras, polynomial periodic cyclic homology is the same as periodic cyclic homology. As a consequence, polynomial periodic cyclic homology of commutative algebras acquires a filtration whose associated graded pieces are equivalent to shifts of (non-Hodge-completed) derived de Rham cohomology. We then define a “topological” analogue (Definition 4.9), and for smooth algebras over perfectoid rings, it acquires a *motivic filtration* (Construction 4.13) whose associated graded pieces are equivalent to shifts of (non-Nygaard-completed) prismatic cohomology.

We first give some sufficient conditions for assembly maps being equivalences.

**Lemma 4.1.** Let  $A$  is a commutative algebra in cyclonic spectra (resp.  $G$ -spectra for a finite group  $G$ ) whose underlying Borel equivariant spectrum, denoted by  $A^h$ , is  $t$ -bounded. Then the assembly map  $(-)^{\theta_A G} \rightarrow (-)^{tG}$  (resp.  $(-)^{\eta_A G} \rightarrow (-)^{hG}$ ) is an equivalence on the idempotent-complete stable subcategory of  $\mathrm{Mod}_A(\mathrm{Sp}^{g^<\mathbb{T}})$  (resp.  $\mathrm{Mod}_A(\mathrm{Sp}^{g^G})$ ) generated by  $A$ -modules of the form  $\bigoplus_{i \in I} A \otimes [\mathbb{T}/H_i]$  (resp.  $\bigoplus_{i \in I} A \otimes [G/H_i]$ ) for an indexed family  $(H_i)_{i \in I}$  of finite cyclic groups  $H_i$  (resp. finite groups  $H_i \subseteq G$ ).

**Proof.** We write the argument for the finite group case. The cyclonic case is similar.

- By construction, the assembly map is an equivalence on  $A \otimes [G/H]$  for finite groups  $H \subseteq G$ .
- The family  $\{A^h \otimes [G/H] \in \mathrm{Sp}^{BG} \mid (H \subseteq G \text{ is a finite subgroup})\}$  is uniformly  $t$ -bounded, thus the canonical map

$$\bigoplus_i (A \otimes [G/H_i])^{hG} \longrightarrow \left( \bigoplus_i A \otimes [G/H_i] \right)^{hG}$$

is an equivalence for an indexed family  $(H_i \subseteq G)_i$  of finite subgroups of  $G$ . It follows that the assembly map is an equivalence on  $\bigoplus_i A \otimes [G/H_i]$ .

- The result follows from the fact that the functors  $(-)^{\theta_A G}$ ,  $(-)^{\eta_G}$ ,  $(-)^{tG}$  and  $(-)^{hG}$  are exact.  $\square$

When  $G$  is a finite cyclic group, the situation is particularly simple.

<sup>3.1.</sup> This means that it is bounded with respect to the canonical  $t$ -structure. We add the prefix “ $t$ ” to avoid confusion with boundedness of  $p$ -power torsion (which we do not use in this article anyways).

**Corollary 4.2.** *Let  $G$  be a finite cyclic group. Then the assembly map  $(-)^{\theta_{\mathbb{Z}} G} \rightarrow (-)^{\theta_{\mathbb{Z}} G}$  (resp.  $(-)^{\eta_{\mathbb{Z}} G} \rightarrow (-)^{\eta_{\mathbb{Z}} G}$ ) is an equivalence on  $t$ -bounded objects in  $\text{Mod}_{\mathbb{Z}}(\text{Sp}^{gG})$ .*

**Proof.** Since the abelian category  $\text{Mack}_G^{\text{coh}}$  has finite projective dimension when  $G$  is finite cyclic [BSW17, Cor 7.2], every  $t$ -bounded object is represented by a finite complex of projective objects. Thus the assembly map  $(-)^{\theta_G} \rightarrow (-)^{tG}$  is an equivalence on these objects by Lemma 4.1.  $\square$

**Corollary 4.3.** *Let  $G$  be a finite cyclic group, and  $A$  a commutative algebra in  $D^b \text{Mack}_G^{\text{coh}} \subseteq \text{Mod}_{\mathbb{Z}}(\text{Sp}^{gG})$ . Then the assembly map  $(-)^{\theta_{\mathbb{Z}} G} \rightarrow (-)^{\theta_A G}$  (resp.  $(-)^{\eta_{\mathbb{Z}} G} \rightarrow (-)^{\eta_A G}$ ) is an equivalence. Consequently, the  $A$ -de-completed  $G$ -Tate construction  $(-)^{\theta_A G}$  (resp. the  $A$ -de-completed homotopy  $G$ -fixed points  $(-)^{\eta_A G}$ ) coincides with the composite functor*

$$\text{Mod}_A(\text{Sp}^{gG}) \longrightarrow \text{Mod}_{\mathbb{Z}}(\text{Sp}^{gG}) \xrightarrow{(-)^{\theta_{\mathbb{Z}} G} \text{ or } (-)^{\eta_{\mathbb{Z}} G}} D(\mathbb{Z}).$$

**Proof.** Note that the  $\infty$ -category  $\text{Mod}_A(\text{Sp}^{gG})$  is generated by objects of the form  $M \otimes_{\mathbb{Z}}^L A$  for finite permutation  $G$ -modules  $M$ , which is  $t$ -bounded since  $M$  is  $\mathbb{Z}$ -flat, and the result follows from Corollary 4.2.  $\square$

We are ready to establish the comparison between the polynomial periodic (resp. negative) cyclic homology and the periodic (resp. negative) cyclic homology on smooth algebras:

**Proposition 4.4.** *Let  $k$  be a  $t$ -bounded animated ring, and  $R$  an  $\mathbb{E}_1$ - $k$ -algebra. Then the commutative diagram*

$$\begin{array}{ccc} \text{HC}^{-, \text{poly}}(R/k) & \longrightarrow & \text{HC}^{-}(R/k) \\ \downarrow & & \downarrow \\ \text{HP}^{\text{poly}}(R/k) & \longrightarrow & \text{HP}(R/k) \end{array}$$

*as an instance of (2.1) is Cartesian. If  $R$  is  $p$ -completely smooth as an  $\mathbb{E}_1$ - $k$ -algebra, and have bounded Tor-amplitude in  $D(k)_p^{\wedge}$ , then the horizontal assembly maps are equivalences after  $p$ -completion.*

We first give a proof in the special case of  $R$  being a  $p$ -completely smooth animated  $k$ -algebra (i.e. with commutativity), since the general case needs knowledge on polygonic spectra in [KM23], and the commutative case is sufficient for this section.

**Proof of Proposition 4.4 with commutativity.** By Remark 2.2, this commutative diagram is Cartesian, thus for  $p$ -completely smooth animated  $k$ -algebras  $R$ , it suffices to check that the map  $\text{HP}^{\text{poly}}(R/k) \rightarrow \text{HP}(R/k)$  is an equivalence after modulo  $p$ , which is equivalent to base change along  $k \rightarrow k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p$ , thus we may assume that  $k$  is a  $t$ -bounded animated  $\mathbb{F}_p$ -algebra. We can further check it after modulo  $p$  again. By Remark 2.8, it reduces to check that the assembly map

$$\text{HH}(R/\underline{k})^{\theta_{\underline{k}} C_p} \longrightarrow \text{HH}(R/k)^{tC_p}$$

is an equivalence. By Corollaries 4.2 and 4.3, it suffices to check that the cyclonic spectrum  $\text{HH}(R/\underline{k}) \in \text{Mod}_{\underline{k}}(\text{Sp}^{g^{<\mathbb{T}}})$  is  $t$ -bounded (thus so after forgetting to  $\text{Sp}^{gC_p}$ ). We check in two steps.

**Polynomial case.** When  $R = P \otimes_{\mathbb{F}_p}^{\mathbb{L}} k$  where  $P$  is a (finite) polynomial  $\mathbb{F}_p$ -algebra. Then we have  $\text{HH}(R/\underline{k}) = \text{HH}(P/\mathbb{F}_p) \otimes_{\mathbb{F}_p}^{\mathbb{L}} k$ , and by  $t$ -boundedness of  $k$ , it suffices to show that  $\text{HH}(P/\mathbb{F}_p)$  is  $t$ -bounded. This follows from [Hes96, 2.2.4 & 2.2.5].

**General case.** By passing to a Zariski cover, we may assume that there exists an étale map  $S \rightarrow R$  where  $S$  is a (finite) polynomial  $k$ -algebra. Then by [HLL20, Add 3.2] (along with [Bor11, Thm B], which is used in their proof), the map  $\text{THH}(S) \rightarrow \text{THH}(R)$  is flat (even étale) in  $\text{CAlg}(\text{Sp}^{g^{<\mathbb{T}}})$ , thus so is the map  $\text{HH}(S/\underline{k}) \rightarrow \text{HH}(R/\underline{k})$ , and the result follows.  $\square$

Now we give the proof for Proposition 4.4 in full generality. As in the first proof of Proposition 4.4, it reduces to check that the assembly map

$$\text{HH}(R/\underline{k})^{\theta_{\underline{k}} C_p} \longrightarrow \text{HH}(R/k)^{tC_p}$$

is an equivalence when  $k$  is a  $t$ -bounded animated  $\mathbb{F}_p$ -algebra, and  $R$  is  $p$ -completely smooth as an  $\mathbb{E}_1$ - $k$ -algebra with bounded Tor-amplitude in  $D(k)$ . We prove a generalized version with coefficients.

Let  $R$  be an  $\mathbb{E}_1$ -ring spectrum, and  $M$  an  $R$ - $R$ -bimodule in  $\mathrm{Sp}$ . Then by [KM23, §6], the topological Hochschild homology  $\mathrm{THH}(R; M)$  carries a canonical  $p$ -polygonic structure, given by the sequence  $(\mathrm{THH}(R; M^{\otimes_{\mathbb{F}_p}^r}) \in \mathrm{Sp}^{BC_{p^r}})_{r \in \mathbb{N}}$ , along with polygonic Frobenius maps

$$\mathrm{THH}(R; M^{\otimes_{\mathbb{F}_p}^r}) \longrightarrow \mathrm{THH}(R; M^{\otimes_{\mathbb{F}_p}^{r+1}})^{tC_p}$$

in  $\mathrm{Sp}^{BC_{p^r}}$ . Moreover, when all  $\mathrm{THH}(R; M^{\otimes_{\mathbb{F}_p}^r})$  in question are bounded below (this is the case when  $M$  is a perfect  $R$ - $R$ -bimodule in  $\mathrm{Sp}$ ), we get a sequence  $(\mathrm{THH}(R; M^{\otimes_{\mathbb{F}_p}^r}) \in \mathrm{Sp}^{gC_{p^r}})_{r \in \mathbb{N}}$  with equivalences

$$\mathrm{THH}(R; M^{\otimes_{\mathbb{F}_p}^{r+1}})^{\Phi C_p} \xrightarrow{\simeq} \mathrm{THH}(R; M^{\otimes_{\mathbb{F}_p}^r})$$

of genuine  $C_{p^r}$ -spectra. This construction extends to the case without bounded-below-ness, by a forthcoming work by Harpaz–Nikolaus–Saunier. All of these constructions are functorial in  $(R; M)$ .

There is also a forgetful functor from cyclotomic spectra to polygonic spectra, and  $\mathrm{THH}(R; R)$  as a polygonic spectrum is the same as the underlying polygonic spectrum of the cyclotomic spectrum  $\mathrm{THH}(R)$ .

**Construction 4.5.** Let  $k$  be an animated ring,  $R$  an  $\mathbb{E}_1$ - $k$ -algebra, and  $M$  an  $R$ - $R$ -bimodule in  $D(k)$ . Then for every  $r \in \mathbb{N}$ , we get a genuine equivariant enhancement  $\mathrm{HH}((R; M)/\underline{k})$  of Hochschild homology  $\mathrm{HH}((R; M^{\otimes_{\mathbb{F}_p}^r})/k)$  with coefficients given by

$$\mathrm{HH}((R; M^{\otimes_{\mathbb{F}_p}^r})/\underline{k}) := \mathrm{THH}(R; M^{\otimes_{\mathbb{F}_p}^r}) \otimes_{\mathrm{THH}(k)}^{\mathbb{L}} \underline{k}.$$

**Lemma 4.6.** *Let  $k$  be a  $t$ -bounded animated ring,  $R$  an  $\mathbb{E}_1$ - $k$ -algebra with bounded Tor-amplitude in  $D(k)$ , and  $M$  a perfect  $R$ - $R$ -bimodule in  $D(k)$ . Then the assembly map*

$$\mathrm{HH}((R; M^{\otimes_{\mathbb{F}_p}^r})/\underline{k})^{\theta_k C_p} \longrightarrow \mathrm{HH}((R; M^{\otimes_{\mathbb{F}_p}^r})/k)^{tC_p}$$

*is an equivalence.*

**Proof.** The proof of [NS18, Prop III.1.1] (or [Lur11, Prop 2.2.3]) implies that the functor  $\mathrm{HH}((R; (-)^{\otimes_{\mathbb{F}_p}^r})/k)^{tC_p}$  is exact. By Lemma 2.10, the symmetric monoidal structure on  $(-)^{\Phi C_p}$ , and the equivalence  $\mathrm{THH}(R; M^{\otimes_{\mathbb{F}_p}^r})^{\Phi C_p} \xrightarrow{\simeq} M$ , we see that the functor  $\mathrm{HH}((R; (-)^{\otimes_{\mathbb{F}_p}^r})/\underline{k})^{\Phi C_p}$  is exact as well. Thus, to see that the assembly map in question is an equivalence, it suffices to show that it is an equivalence when  $M = R \otimes_k^{\mathbb{L}} R$ , the free  $R$ - $R$ -bimodule in  $D(k)$  of rank 1. In this case, for every  $r \in \mathbb{N}$ , we have an equivalence

$$\mathrm{THH}(R; (R \otimes_k^{\mathbb{L}} R)^{\otimes_{\mathbb{F}_p}^r}) \simeq \mathrm{THH}(k; R^{\otimes_{\mathbb{F}_p}^r})$$

in  $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{gC_{p^r}})$ , and after base change along  $\mathrm{THH}(k) \rightarrow \underline{k}$ , it becomes the relative  $C_{p^r}$ -norm  $R^{\otimes_k^{\mathbb{L}} C_{p^r}}$ . Under this identification, the assembly map in question becomes the assembly map

$$(R^{\otimes_k^{\mathbb{L}} C_p})^{\theta_k C_p} \longrightarrow (R^{\otimes_k^{\mathbb{L}} C_p})^{tC_p},$$

which is an equivalence by Lemma 4.7.  $\square$

**Lemma 4.7.** *Let  $k$  be a  $t$ -bounded animated ring, and  $M$  a  $k$ -module spectrum of bounded Tor-amplitude in  $D(k)$ . Then the assembly map*

$$(M^{\otimes_k^{\mathbb{L}} C_p})^{\theta_k C_p} \longrightarrow (M^{\otimes_k^{\mathbb{L}} C_p})^{tC_p}$$

*is an equivalence.*

**Proof.** Since both sides are exact in  $M$ , we may assume that  $M$  is a flat  $k$ -module spectrum, which is a filtered colimit  $\mathrm{colim}_i M_i$  of finite free  $k$ -module spectra by Lazard’s theorem [Lur17, Thm 7.2.2.15]. In this case, each derived cohomological  $C_{p^r}$ -Mackey functor  $M_i^{\otimes_k^{\mathbb{L}} C_p}$  is  $t$ -bounded by the  $t$ -boundedness of  $k$ , thus so is their filtered colimit  $M^{\otimes_k^{\mathbb{L}} C_p}$ . The result follows from Corollaries 4.2 and 4.3.  $\square$

**Proof of Proposition 4.4 in general.** It follows from Lemma 4.6 by setting  $M = R$ .  $\square$

**Question 2.** Is there any categorical generalization of Lemma 4.6, namely, replacing  $R$  by dualizable presentable stable  $k$ -linear  $\infty$ -category  $\mathcal{C}$  which is smooth with bounded Tor-amplitude<sup>4.1</sup>, and replacing  $M$  by an colimit-preserving  $k$ -linear endofunctor  $\mathcal{C} \rightarrow \mathcal{C}$ ?

Now we construct an HKR filtration on the polynomial periodic (resp. negative) cyclic homology whose associated graded pieces are shifts of (non-Hodge-completed) derived de Rham cohomology (resp. its Hodge-filtered pieces), Hodge-de-completing [BMS19, Thm 1.17] and [Ant19].

**Construction 4.8. (HKR filtration)** Let  $k$  be a  $t$ -bounded animated ring. Then by [Rak20] (which generalizes [Ant19]), for every smooth  $k$ -algebra  $R$ , there is an exhaustive filtration  $\mathrm{Fil}_{\mathrm{HKR}}^*$ , functorial in  $R$ , on the canonical  $p$ -completed map

$$\mathrm{HC}^-(R/k)_p^\wedge \longrightarrow \mathrm{HP}(R/k)_p^\wedge$$

whose  $i$ -th associated graded piece  $\mathrm{gr}_{\mathrm{HKR}}^i$  is given by the canonical map

$$\mathrm{Fil}_H^i \mathrm{dR}_{R/k}[2i]_p^\wedge \rightarrow \mathrm{dR}_{R/k}[2i]_p^\wedge.$$

By Proposition 4.4 and sifted-colimit-preservation of the functors  $\mathrm{CAlg}_k^{\mathrm{an}} \rightarrow D(k)$ ,  $R \mapsto \mathrm{HP}^{\mathrm{poly}}(R/k)$  (resp.  $R \mapsto \mathrm{HC}^{-, \mathrm{poly}}(R/k)$ ), for every animated  $k$ -algebra  $R$ , we get an exhaustive filtration  $\mathrm{Fil}_{\mathrm{HKR}}^*$  on the Cartesian square

$$\begin{array}{ccc} \mathrm{HC}^{-, \mathrm{poly}}(R/k)_p^\wedge & \longrightarrow & \mathrm{HC}^-(R/k)_p^\wedge \\ \downarrow & & \downarrow \\ \mathrm{HP}^{\mathrm{poly}}(R/k)_p^\wedge & \longrightarrow & \mathrm{HP}(R/k)_p^\wedge \end{array}$$

whose  $i$ -th associated graded piece  $\mathrm{gr}_{\mathrm{HKR}}^i$  is given by

$$\begin{array}{ccc} \mathrm{Fil}_H^i \mathrm{dR}_{R/k}[2i]_p^\wedge & \longrightarrow & \mathrm{Fil}_H^i \widehat{\mathrm{dR}}_{R/k}[2i]_p^\wedge \\ \downarrow & & \downarrow \\ \mathrm{dR}_{R/k}[2i]_p^\wedge & \longrightarrow & \widehat{\mathrm{dR}}_{R/k}[2i]_p^\wedge \end{array}$$

where  $\widehat{\mathrm{dR}}_{R/k}$  is the Hodge-completed derived de Rham cohomology of  $R/k$ .

**Question 3.** Is the HKR filtration in Construction 4.8 complete?

It is very natural to extend our definition to de-Nygaard-complete topological periodic cyclic homology.

**Definition 4.9.** Let  $k$  be a  $t$ -bounded  $\mathbb{E}_\infty$ -ring, and  $\mathcal{C}$  a dualizable presentable stable  $k$ -linear  $\infty$ -category. Then the topological  $k$ -polynomial periodic cyclic homology  $\mathrm{TP}^{\mathrm{poly}/k}(\mathcal{C})$  (resp. the topological  $k$ -polynomial negative cyclic homology  $\mathrm{TC}^{-, \mathrm{poly}/k}(\mathcal{C})$ ) is defined to be the  $\mathrm{THH}(k)$ -de-completed  $\mathbb{T}$ -Tate construction  $\mathrm{THH}(\mathcal{C})^{\theta_{\mathrm{THH}(k)} \mathbb{T}}$  (resp. the  $\mathrm{THH}(k)$ -de-completed homotopy  $\mathbb{T}$ -fixed points  $\mathrm{THH}(\mathcal{C})^{\eta_{\mathrm{THH}(k)} \mathbb{T}}$ ).

Topological  $k$ -polynomial periodic (resp. negative) cyclic homology, even after  $p$ -completion, seems intractable in general, partially due to the global nature of its prismatization. The situation is drastically simpler when the ring  $k = S$  is  $p$ -complete and perfectoid, thanks to the Bökstedt periodicity of  $\mathrm{THH}(S)_p^\wedge$ . We recollect some notations and computations in [BMS19, Prop 6.2 & 6.3]. However, we view  $\mathrm{THH}(S)^{tC_p}$  non-equivariantly as a  $\mathrm{TC}^-(S)$ -module, which follows more closely to the convention in [Rig25, Lem 2.1].

<sup>4.1</sup> An attempt for this definition: a dualizable presentable stable  $k$ -linear  $\infty$ -category  $\mathcal{C}$  has bounded Tor-amplitude if the coevaluation functor  $\mathcal{C}^\vee \otimes_{D(k)} \mathcal{C} \rightarrow D(k)$  sends compact objects to objects of bounded Tor-amplitude in  $D(k)$ .

**Remark 4.10.** ([BMS19, Prop 6.2 & 6.3]) Let  $S$  be a perfectoid ring,  $A := A_{\text{inf}}(S) = W(S^\flat)$  with Frobenius endomorphism  $\varphi: A \rightarrow A$ , and  $\xi$  a chosen generator of the kernel  $\ker(\theta)$  of Fontaine's map  $\theta: A \rightarrow S$ . Then the commutative square

$$\begin{array}{ccc} \text{TC}^-(S)_p^\wedge & \xrightarrow{\varphi_p^{h\mathbb{T}}} & \text{TP}(S)_p^\wedge \\ \downarrow & & \downarrow \\ \text{THH}(S)_p^\wedge & \xrightarrow{\varphi_p} & \text{THH}(S)^{tC_p} \end{array}$$

is a pushout diagram of  $\mathbb{E}_\infty$ -rings, and its homotopy groups are given by

$$\begin{array}{ccc} A[u, v] / (uv - \xi) & \xrightarrow[u \mapsto \varphi(\xi)\sigma^{-1}]{u \mapsto \sigma} & A[\sigma^\pm] \\ \downarrow & & \downarrow \\ R[u] = (A/\xi)[u] & \xrightarrow{u \mapsto \sigma} & (A/\varphi(\xi))[\sigma^\pm] \end{array},$$

where  $|u| = |\sigma| = 2$  and  $|v| = -2$ , the vertical maps are  $A$ -linear, and the horizontal maps are  $\varphi$ -linear. The homotopy groups of the canonical map  $\text{TC}^-(S)_p^\wedge \rightarrow \text{TP}(S)_p^\wedge$  is given by the  $A$ -linear map

$$A[u, v] / (uv - \xi) \xrightarrow[u \mapsto \sigma^{-1}]{u \mapsto \xi\sigma} A[\sigma^\pm].$$

In particular, the  $\text{THH}(S)$ -module  $S = \text{THH}(S)/u$  in  $\text{Sp}^{B\mathbb{T}}$  is perfect, which implies that

**Lemma 4.11.** *Let  $S$  be a perfectoid ring, and  $M$  a  $\text{THH}(S)$ -module in  $\text{Sp}^{g < \mathbb{T}}$ . Then the canonical map*

$$M^{\theta_{\text{THH}(S)}\mathbb{T}} \otimes_{\text{TP}(S)}^{\mathbb{L}} S^{t\mathbb{T}} \longrightarrow (M \otimes_{\text{THH}(S)} S)^{\theta_S\mathbb{T}}$$

is an equivalence after  $p$ -completion.

**Corollary 4.12.** *Let  $S$  be a perfectoid ring, and  $\mathcal{C}$  a dualizable presentable stable  $S$ -linear  $\infty$ -category. Then the commutative diagram*

$$\begin{array}{ccc} \text{TC}^{-, \text{poly}/S}(\mathcal{C}) & \longrightarrow & \text{TC}^-(\mathcal{C}) \\ \downarrow & & \downarrow \\ \text{TP}^{\text{poly}/S}(\mathcal{C}) & \longrightarrow & \text{TP}(\mathcal{C}) \end{array}$$

as an instance of (2.1) is Cartesian. If the assembly map  $\text{HP}^{\text{poly}}(\mathcal{C}/S) \rightarrow \text{HP}(\mathcal{C}/S)$  is an equivalence after  $p$ -completion<sup>4.2</sup>, then the horizontal assembly maps are equivalences after  $(p, \ker(\theta))$ -completion.

**Proof.** By Remark 2.2, this commutative diagram is Cartesian. When the assembly map  $\text{HP}^{\text{poly}}(\mathcal{C}/S) \rightarrow \text{HP}(\mathcal{C}/S)$  is  $p$ -completely an equivalence, then by Remark 4.10, it is the  $(\text{mod } \ker(\theta))$  reduction of the map  $\text{TP}^{\text{poly}/S}(\mathcal{C}) \rightarrow \text{TP}(\mathcal{C})$ , thus the later is  $(p, \ker(\theta))$ -completely an equivalence, and the result follows.  $\square$

Similarly to Construction 4.8, we have

**Construction 4.13. (Motivic filtration)** Let  $S$  be a perfectoid ring, and let  $A := A_{\text{inf}}(S)$ . Then by [BMS19], for every smooth  $S$ -algebra  $R$ , there is an exhaustive filtration  $\text{Fil}_M^*$ , functorial in  $R$ , on the canonical  $(p, \ker(\theta))$ -completed<sup>4.3</sup> map

$$\text{TC}^-(R)_{(p, \ker(\theta))}^\wedge \longrightarrow \text{TP}(R)_{(p, \ker(\theta))}^\wedge$$

<sup>4.2</sup>. By Proposition 4.4, this is the case when  $\mathcal{C} = D(R)$  for some  $p$ -completely smooth  $\mathbb{E}_1$ - $S$ -algebra  $R$ .

<sup>4.3</sup>. The  $p$ -completed  $\text{TC}^-(R)_p^\wedge$  and  $\text{TP}(R)_p^\wedge$  are automatically  $\ker(\theta)$ -complete (since they are Nygaard-complete), but it is conceptually better to phrase it after  $(p, \ker(\theta))$ -completion, since the  $p$ -completed polynomial versions might not be  $\ker(\theta)$ -complete.

whose  $i$ -th associated graded piece  $\mathrm{gr}_M^i$  is given by the canonical map

$$\mathrm{Fil}_N^i \varphi_A^* \Delta_{R/A} \longrightarrow \varphi_A^* \Delta_{R/A}$$

for the Frobenius twisted prismatic cohomology. By Proposition 4.4 and Corollary 4.12, and sifted-colimit-preservation of the functors  $\mathrm{CAlg}^{\mathrm{an}} \rightarrow D(\mathrm{TC}^-(S))$ ,  $R \mapsto \mathrm{TP}^{\mathrm{poly}/S}(R)$  (resp.  $R \mapsto \mathrm{TC}^{-,\mathrm{poly}/S}(R)$ ), for every animated  $k$ -algebra  $R$ , we get an exhaustive filtration  $\mathrm{Fil}_M^*$  on the Cartesian square

$$\begin{array}{ccc} \mathrm{TC}^{-,\mathrm{poly}/S}(R)_{(p,\ker(\theta))}^\wedge & \longrightarrow & \mathrm{TC}^-(R)_{(p,\ker(\theta))}^\wedge \\ \downarrow & & \downarrow \\ \mathrm{TP}^{\mathrm{poly}/S}(R)_{(p,\ker(\theta))}^\wedge & \longrightarrow & \mathrm{TP}(R)_{(p,\ker(\theta))}^\wedge \end{array}$$

whose  $i$ -th associated graded piece is given by

$$\begin{array}{ccc} \mathrm{Fil}_N^i \varphi_A^* \Delta_{R/A} & \longrightarrow & \mathrm{Fil}_N^i \varphi_A^* \hat{\Delta}_{R/A} \\ \downarrow & & \downarrow \\ \varphi_A^* \Delta_{R/A} & \longrightarrow & \varphi_A^* \hat{\Delta}_{R/A} \end{array}.$$

**Remark 4.14.** The construction of HKR filtration and motivic filtration on polynomial cyclic theories is quite formal: one only needs a proposition similar to Proposition 4.4 and Corollary 4.12. In particular, the above construction also adapts to the Breuil–Kisin case (as in [BMS19, §11]) and the  $q$ -de Rham case.

**Remark 4.15.** The topological polynomial periodic cyclic homology should be comparable to  $\mathrm{TC}^{(-1)}$  as introduced in [Man24, §1]. In Section 5, we will discuss and compare to the construction in the absolute case in an ongoing project of Devalapurkar–Hahn–Raksit–Yuan.

A natural question is whether the previous picture extends to relative prismatic cohomology over an arbitrary base prism? When the base prism is transversal, we have the following expectation.

**Remark 4.16.** In a companion paper [Mao24b], we defined *prismatic Hochschild homology*  $\mathrm{HH}^\Delta(\mathcal{C}/A)$  for a transversal prism  $(A, I)$  and a dualizable presentable stable  $A/I$ -linear  $\infty$ -category  $\mathcal{C}$ , and formulated an HKR-type conjecture for  $p$ -completely smooth  $A/I$ -algebras. If that conjecture holds, then the  $p$ -completed  $(A, I)^{C_{p^{r-1}}}$ -de-completed  $\mathbb{T}/C_{p^{r-1}}$ -construction  $(\mathrm{HH}^\Delta(R/A)^{C_{p^{r-1}}})_{(A, I)^{C_{p^{r-1}}}}^{\theta}(\mathbb{T}/C_{p^{r-1}})$  for animated  $(A/I)$ -algebra would carry an exhaustive filtration with associated graded pieces equivalent to shifts of  $(\varphi_A^* \Delta_{R/A}) \otimes_A^{\mathbb{L}} (A/I_r)$ . We will address this in the future.

Up to our knowledge, it was not widely expected that the topological Hochschild homology as a cyclotomic spectrum contains enough information for a de-Nygaard-completion such as Definition 4.9. However, in view of Efimov’s rigidity of localizing motives, this is expected:

**Remark 4.17. (M. RAMZI)** Let  $k$  be a commutative ring, and  $G$  a finite group,  $A$  an  $\mathbb{E}_\infty$ -algebra in  $\mathrm{Sp}^{gG}$ , and  $E: \mathrm{Cat}_k^{\mathrm{perf}} \rightarrow \mathrm{Mod}_A(\mathrm{Sp}^{BG})$  a finitary symmetric monoidal localizing invariant. Then the symmetric monoidal functor  $E$  factors uniquely through the presentably stable symmetric monoidal  $\infty$ -category  $\mathrm{Mot}_{\mathrm{loc},k}$ , obtaining a functor  $\mathrm{Mot}_{\mathrm{loc},k} \rightarrow \mathrm{Mod}_A(D(k)^{BG})$  in  $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{St}}^L)$ . Efimov’s rigidity theorem, as mentioned in [Efi24, Rem 4.3], tells us that the presentably stable symmetric monoidal  $\infty$ -category  $\mathrm{Mot}_{\mathrm{loc},k}$  is rigid, thus we get a unique strongly continuous functor  $\mathrm{Mot}_{\mathrm{loc},k} \rightarrow \mathrm{Mod}_A(D(k)^{BG})^{\mathrm{rig}}$ .

If we are in addition given a finitary symmetric monoidal factorization

$$\mathrm{Cat}_k^{\mathrm{perf}} \xrightarrow{\hat{E}} \mathrm{Mod}_A(\mathrm{Sp}^{gG}) \longrightarrow \mathrm{Mod}_A(D(k)^{BG})$$



of  $E$ , where the functor  $\tilde{E}$  is localizing as well, then by Remark 2.7, we see that the strongly continuous functor  $\mathrm{Mot}_{\mathrm{loc},k} \rightarrow \mathrm{Mod}_A(D(k)^{BG})^{\mathrm{rig}}$  coincides with the composite functor

$$\mathrm{Mot}_{\mathrm{loc},k} \xrightarrow{\tilde{E}} \mathrm{Mod}_A(\mathrm{Sp}^{gG}) \longrightarrow \mathrm{Mod}_A(D(k)^{BG})^{\mathrm{rig}}.$$

Informally, this tells us that  $\tilde{E}$  “knows” everything about refined  $E$ .

Now we apply this to the finitary symmetric monoidal localizing invariant

$$\tilde{E} := \mathrm{THH} : \mathrm{Cat}_k^{\mathrm{perf}} \longrightarrow \mathrm{CycSp}^{\mathrm{gen}} \longrightarrow \mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{gG})$$

for any finite cyclic group  $G$ . It follows that the refinement of  $\mathrm{Sp}^{BG}$ -valued THH is completely determined by the functor  $\tilde{E}$ , thus by the (genuine) cyclotomic THH. Roughly speaking, this implies that the cyclotomic THH already contains “all” profinite or  $p$ -adic formal information, including any de-Nygaard-completion.

## 5 The absolute case

In an ongoing work of Devalapurkar–Hahn–Raksit–Yuan, they propose another de-completion  $\mathrm{THH}_{\Delta}$  of topological negative cyclic homology for animated rings. In this section, we show that our techniques apply to this situation as well, which leads to *de-completed Borel completion* (Definition 5.4), giving rise to a localizing invariant on  $\mathbb{Z}$ -linear categories corresponding to *absolute prismatic cohomology*. We also compare de-completed Borel completion with de-completed homotopy fixed points and de-completed Tate construction (Remarks 5.7 and 5.8, Proposition 5.23, and Corollary 5.26), and the construction in this section with previous constructions (Corollaries 5.31 and 5.35). We first recall their definition. Actually, our argument shows that  $\mathrm{THH}_{\Delta}$  extends to a localizing invariant for  $\mathrm{BP}\langle n \rangle$ -linear categories Remark 5.32, which is independent of choice of  $n$ .

**Notation 5.1.** *Let  $G$  be a finite group (resp.  $\mathbb{T}$ ). We will denote by*

$$(-)^h : \mathrm{Sp}^{gG} \longrightarrow \mathrm{Sp}^{gG}$$

*(resp.*

$$(-)^h : \mathrm{Sp}^{g<\mathbb{T}} \longrightarrow \mathrm{Sp}^{g<\mathbb{T}}$$

*) the Borel completion, which has a lax symmetric monoidal structure.*

**Construction 5.2. (Devalapurkar–Hahn–Raksit–Yuan)** The functor

$$\mathrm{THH}_{\Delta} : \mathrm{CAlg}_{\mathbb{Z}}^{\mathrm{an}} \longrightarrow \mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})^h}(\mathrm{Sp}^{g<\mathbb{T}})$$

is defined to be the left derived functor of the functor

$$\begin{aligned} \mathrm{Poly}_{\mathbb{Z}}^{\mathrm{fg}} &\longrightarrow \mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})^h}(\mathrm{Sp}^{g<\mathbb{T}}) \\ R &\longmapsto \mathrm{THH}(R)^h. \end{aligned}$$

By construction, we have a canonical map

$$\mathrm{THH}_{\Delta}(R) \longrightarrow \mathrm{THH}(R)^h$$

in  $\mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})^h}(\mathrm{Sp}^{g<\mathbb{T}})$ , which is an equivalence when  $R$  is a finitely generated polynomial ring.

**Remark 5.3. (Devalapurkar–Hahn–Raksit–Yuan)** It follows immediately from the discussion of [BL22, §6.2] (which explains the result in [BMS19] for animated rings) that, there exists a motivic filtration on  $\mathrm{THH}(R)^{C_{p^\infty}}$  (resp.  $(\mathrm{THH}(R)^{\Phi C_p})^{C_{p^\infty}/C_p}$ ) whose associated graded pieces are given by twisted absolute Nygaard cohomology  $\mathrm{Fil}_N^* \Delta_R\{*\}[2*]$  (resp. twisted absolute prismatic cohomology  $\Delta_R\{*\}[2*]$ ).

To extend  $\mathrm{THH}_{\Delta}$  to a localizing invariant, as in Definition 3.3, the key is to de-complete Borel completion.

**Definition 5.4.** Let  $G$  be a finite group, and  $A$  an  $\mathbb{E}_1$ -algebra in the symmetric monoidal  $\infty$ -category  $\mathrm{Sp}^{gG}$  of  $G$ -spectra. Then the  $A$ -de-completed Borel completion, denoted by  $(-)^{\eta_A}$ , is the filtered-colimit-preserving approximation

$$\mathrm{RMod}_A(\mathrm{Sp}^{gG}) \longrightarrow \mathrm{RMod}_{A^h}(\mathrm{Sp}^{gG})$$

of the Borel completion. The same if we replace  $G$  by  $\mathbb{T}$ , and the symmetric monoidal  $\infty$ -category  $\mathrm{Sp}^{gG}$  of  $G$ -spectra by that  $\mathrm{Sp}^{g<\mathbb{T}}$  of cyclonic spectra.

**Remark 5.5.** Let  $G$  be a finite group (resp.  $\mathbb{T}$ ), and  $A$  an  $\mathbb{E}_1$ -algebra in the symmetric monoidal  $\infty$ -category  $\mathrm{Sp}^{gG}$  (resp.  $\mathrm{Sp}^{g<\mathbb{T}}$ ) of  $G$ -spectra (resp. cyclonic spectra). As in Remark 2.4, the  $A$ -de-completed Borel completion factors canonically through the base change along the Borel completion map  $A \rightarrow A^h$ .

Actually, the de-completed Borel completion is simply a base change.

**Lemma 5.6.** Let  $G$  be a finite group, and  $A$  an  $\mathbb{E}_1$ -algebra in the symmetric monoidal  $\infty$ -category  $\mathrm{Sp}^{gG}$  of  $G$ -spectra. Then the  $A$ -de-completed Borel completion  $(-)^{\eta_A}$  coincides with the base change functor

$$(-) \otimes_A^{\mathbb{L}} A^h : \mathrm{RMod}_A(\mathrm{Sp}^{gG}) \longrightarrow \mathrm{RMod}_{A^h}(\mathrm{Sp}^{gG}).$$

The same if we replace  $G$  by  $\mathbb{T}$ , and  $\mathrm{Sp}^{gG}$  by  $\mathrm{Sp}^{g<\mathbb{T}}$ .

**Proof.** We start with the case that  $G$  is a finite group. By Remark 5.5, without loss of generality, we may assume that  $A$  is Borel complete, and by definition, it suffices to check that, for every  $G$ -orbit  $[G/H]$ , the tensor product  $[G/H] \otimes A$  is Borel complete. This follows from the dualizability of  $[G/H]$  in  $\mathrm{Sp}^{gG}$ , cf. the proof of [BCN21, Ex 2.15].

The case for  $G = \mathbb{T}$  is similar: we have to show that  $[\mathbb{T}/C_n] \otimes A$  is Borel complete, and it suffices to restrict to  $\mathrm{Sp}^{gC_m}$  for every  $m \in \mathbb{N}_{>0}$ , and  $[\mathbb{T}/C_n]$  is dualizable there as well.  $\square$

It then follows from Remark 2.4 that de-completed homotopy fixed points and de-completed Tate construction factor through de-completed Borel completion, thus the latter contains more information than the former. When the group is finite, this factorization takes a very simple form:

**Remark 5.7.** Let  $G$  be a finite group, and  $A$  an  $\mathbb{E}_1$ -algebra in the symmetric monoidal  $\infty$ -category  $\mathrm{Sp}^{gG}$  of  $G$ -spectra. By restricting to compact objects in  $\mathrm{RMod}_A(\mathrm{Sp}^{gG})$ , we see that the  $A$ -de-completed homotopy  $G$ -fixed points  $(-)^{\eta_A G}$  factors as

$$\mathrm{RMod}_A(\mathrm{Sp}^{gG}) \xrightarrow{(-)^{\eta_A}} \mathrm{RMod}_{A^h}(\mathrm{Sp}^{gG}) \xrightarrow{(-)^G} \mathrm{RMod}_{A^h G}(\mathrm{Sp}).$$

**Remark 5.8.** Let  $G$  be a finite group, and  $A$  an  $\mathbb{E}_1$ -algebra in the symmetric monoidal  $\infty$ -category  $\mathrm{Sp}^{gG}$  of  $G$ -spectra. By restricting to compact objects in  $\mathrm{RMod}_A(\mathrm{Sp}^{gG})$ , we see that the  $A$ -de-completed  $G$ -Tate construction  $(-)^{\eta_A G}$  factors as

$$\mathrm{RMod}_A(\mathrm{Sp}^{gG}) \xrightarrow{(-)^{\eta_A}} \mathrm{RMod}_{A^h}(\mathrm{Sp}^{gG}) \xrightarrow{(-)^{\Phi_{\{e\}}^G}} \mathrm{RMod}_{A^h G}(\mathrm{Sp}),$$

where  $(-)^{\Phi_{\{e\}}^G}$  is the geometric fixed points with respect to the family  $\{e\}$  of subgroups [AMR21, Def 5.7]. When  $G$  is a cyclic  $p$ -group, we have  $(-)^{\Phi_{\{e\}}^G} = (-)^{\Phi_{C_p}}$ .

Actually, this factorization also holds when  $G = \mathbb{T}$  for certain bases which we establish in Proposition 5.23 and Corollary 5.26. Now we formulate the main comparison theorem

**Theorem 5.9.** Let  $R$  be a smooth  $\mathbb{E}_1$ - $\mathbb{Z}$ -algebra of bounded Tor-amplitude<sup>5.1</sup>. Then the assembly map

$$\mathrm{THH}(R)^{\eta_{\mathrm{THH}(\mathbb{Z})}} \longrightarrow \mathrm{THH}(R)^h$$

in  $\mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})^h}(\mathrm{Sp}^{g<\mathbb{T}})$  is an equivalence after  $p$ -completion.

<sup>5.1.</sup> Since the projective dimension of  $\mathbb{Z}$  is finite, this is the same as  $t$ -boundedness.

Before giving its proof, we mention an immediate consequence: since finitely generated polynomial rings are smooth, it follows that

**Corollary 5.10.** *After  $p$ -completion, the functor  $\mathrm{THH}_\Delta$  factors as*

$$\mathrm{CAlg}_{\mathbb{Z}}^{\mathrm{an}} \xrightarrow{\mathrm{Perf}(-)} \mathrm{Cat}_{\mathbb{Z}}^{\mathrm{perf}} \longrightarrow \mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})^h}(\mathrm{Sp}^{g<\mathbb{T}})_p^\wedge,$$

where the second functor is the composite functor

$$\mathrm{Cat}_{\mathbb{Z}}^{\mathrm{perf}} \xrightarrow{\mathrm{THH}} \mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})}(\mathrm{Sp}^{g<\mathbb{T}}) \xrightarrow{((-)^{\eta_{\mathrm{THH}(\mathbb{Z})}})_p^\wedge} \mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})^h}(\mathrm{Sp}^{g<\mathbb{T}})_p^\wedge,$$

being the proposed localizing invariant extending  $\mathrm{THH}_\Delta$ .

The strategy to prove Theorem 5.9 is similar to that for Corollary 4.12. However, the cyclonic spectrum  $\mathrm{THH}(\mathbb{Z})$  is more complicated than  $\mathrm{THH}(S)$  for perfectoid rings  $S$ . Nevertheless, the following two computational inputs, that we learnt from Robert BURKLUND and Jeremy HAHN, are sufficient.

**Lemma 5.11.** ([BM94]) *Let  $p$  be a prime. Then we have*

$$\pi_*(\mathrm{THH}(\mathbb{Z})/p) \cong \mathbb{F}_p[\mu_1] \otimes \Gamma(\lambda_1)$$

with  $\deg \mu_1 = 2p$  and  $\deg \lambda_1 = 2p - 1$ . Moreover, the cyclotomic Frobenius  $\mathrm{THH}(\mathbb{Z}) \rightarrow \mathrm{THH}(\mathbb{Z})^{tC_p}$  realizes  $\pi_*(\mathrm{THH}(\mathbb{Z})^{tC_p}/p)$  as  $\pi_*(\mathrm{THH}(\mathbb{Z})/p)[\mu_1^{-1}]$ .

**Remark 5.12.** More generally, for every  $n \in \mathbb{Z}_{\geq -1}$ , [ACH24, Prop 2.7] shows that, for the  $n$ -th truncated Brown–Peterson spectrum  $\mathrm{BP}\langle n \rangle$ , we have

$$\pi_*(\mathrm{THH}(\mathrm{BP}\langle n \rangle; \mathbb{F}_p)) \cong \mathbb{F}_p[\mu_{n+1}] \otimes \Gamma(\lambda_1, \dots, \lambda_{n+1})$$

where  $\deg \mu_{n+1} = 2p^{n+1}$  and  $\deg \lambda_i = 2p^i - 1$  for  $i = 1, \dots, n+1$ . As explained in the paragraphs following [HRW22, Thm 6.1.2], the cyclotomic Frobenius  $\mathrm{THH}(\mathrm{BP}\langle n \rangle) \rightarrow \mathrm{THH}(\mathrm{BP}\langle n \rangle)^{tC_p}$  realizes the homotopy groups of the target modulo  $(p, v_1, \dots, v_n)$  as the localization of the source modulo  $(p, v_1, \dots, v_n)$  at  $\mu_{n+1}$ .

**Lemma 5.13.** ([BM94]) *Let  $p$  be a prime. Then the class  $\mu_1 \in \pi_{2p}(\mathrm{THH}(\mathbb{Z})/p)$  lifts to a  $\mathbb{T}$ -equivariant class  $\tilde{\mu}_1$  satisfying the relation  $a_\lambda \tilde{\mu}_1 = v_1$ , where  $a_\lambda$  is the Euler class in [BL22, Ex 6.1.3], and  $v_1$  is the canonical class in  $\pi_{2p-2}(\mathrm{TC}^-(\mathbb{Z})/p)^{5.2}$ .*

We give a proof of Proposition 5.15, since we need similar techniques to establish our main results.

**Lemma 5.14.** *Let  $M$  be a  $t$ -bounded  $\mathbb{T}$ -equivariant spectrum. Then for every composite number  $n \in \mathbb{N}_{>0}$ , the proper  $C_n$ -Tate construction<sup>5.3</sup>  $M^{\tau C_n}$  is contractible.*

**Proof.** We may assume that  $M$  is concentrated in degree 0. Then by the lax symmetric monoidal structure on  $(-)^{\tau C_n}$ , we may assume that  $M = \mathbb{Z}$ . Then this is established in [MNN19, Prop 5.24] (cf. [AMR17a, Rem 2.18]).  $\square$

**Proposition 5.15.** (Devalapurkar–Hahn–Raksit–Yuan) *The spectrum  $(\mathrm{THH}(\mathbb{Z})^h)^{\Phi C_n}$  is contractible for any composite number  $n \in \mathbb{N}_{>0}$ .*

**Proof.** Equivalently, we have to show that the proper  $C_n$ -Tate construction  $\mathrm{THH}(\mathbb{Z})^{\tau C_n}$  is contractible for every composite number  $n \in \mathbb{N}_{>0}$ . Fix a prime factor  $p$  of  $n$ , note that it is already  $p$ -complete, equipped with proper Tate diagonal maps

$$\mathrm{THH}(\mathbb{Z}) \longrightarrow \mathrm{THH}(\mathbb{Z})^{\tau C_n}$$

<sup>5.2</sup> When  $p$  is an odd prime, it comes from  $v_1 \in \pi_{2p-2}(\mathbb{S}/p)$ . When  $p = 2$ , over the sphere, only  $v_1^4 \in \pi_8(\mathbb{S}/2)$  is well-defined.

<sup>5.3</sup> See [AMR21, Def 4.5].

as explained in [AMR17b, §0.4], thus  $\mathrm{THH}(\mathbb{Z})^{\tau^{C_n}}$  acquires a  $\mathbb{Z}$ -module structure, and therefore we may check  $\mathrm{THH}(\mathbb{Z})^{\tau^{C_n}} = 0$  after modulo  $(p, v_1)$ . It follows from Lemma 5.14 that

$$(\mathrm{THH}(\mathbb{Z}) / (p, \tilde{\mu}_1))^{\tau^{C_n}} = 0,$$

and by Lemma 5.13, we are done.  $\square$

**Corollary 5.16. (Devalapurkar–Hahn–Raksit–Yuan)** *Let  $M$  be a  $\mathrm{THH}(\mathbb{Z})^h$ -module in  $\mathrm{Sp}^{g < \mathbb{T}}$ . Then the genuine  $C_p$ -fixed points  $M^{C_p}$  and the geometric  $C_p$ -fixed points  $M^{\Phi^{C_p}}$ , viewed as objects in  $\mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})^h C_p}(\mathrm{Sp}^{g < (\mathbb{T}/C_p)})$ , are Borel complete.*

Proposition 5.15 allows us to reduce to analyze de-completed  $C_p$ -Tate construction, by joint conservativity of  $\{(-)^{\Phi^{C_n}} \mid n \in \mathbb{N}_{>0}\}$  and Remark 5.8:

**Corollary 5.17.** *Let  $M$  be a  $\mathrm{THH}(\mathbb{Z})^h$ -module in  $\mathrm{Sp}^{g < \mathbb{T}}$ . Then the assembly map*

$$M^{\eta_{\mathrm{THH}(\mathbb{Z})^h}} \longrightarrow M^h$$

*is an equivalence after  $p$ -completion if and only if the assembly map*

$$M^{\eta_{\mathrm{THH}(\mathbb{Z})^h} C_p} \longrightarrow M^{tC_p}$$

*is an equivalence after  $p$ -completion.*

**Proof of Theorem 5.9.** It follows from Corollary 5.17 and Lemma 5.18.  $\square$

It remains to establish the following version of Lemma 4.6 for  $\mathrm{THH}$ :

**Lemma 5.18.** *Let  $R$  be an  $\mathbb{E}_1$ - $\mathbb{Z}$ -algebra with bounded Tor-amplitude in  $D(\mathbb{Z})$ , and  $M$  a perfect  $R$ - $R$ -bimodule in  $D(\mathbb{Z})$ . Then the assembly map*

$$\mathrm{THH}(R; M^{\otimes_R^{\mathbb{I}} C_p})^{\theta_{\mathrm{THH}(\mathbb{Z})} C_p} \longrightarrow \mathrm{THH}(R; M^{\otimes_R^{\mathbb{I}} C_p})^{tC_p}$$

*is an equivalence after  $p$ -completion.*

**Proof.** We mimick the proof of Lemma 4.6. By the exactness of both sides in  $M$ , we may assume that  $M$  is the free  $R$ - $R$ -bimodule  $R \otimes R$  of rank 1. Then the map in question becomes the assembly map

$$\mathrm{THH}(\mathbb{Z}; R^{\otimes_{\mathbb{Z}}^{\mathbb{I}} p})^{\theta_{\mathrm{THH}(\mathbb{Z})} C_p} \longrightarrow \mathrm{THH}(\mathbb{Z}; R^{\otimes_{\mathbb{Z}}^{\mathbb{I}} C_p})^{tC_p}$$

which follows from Lemma 5.19, being a polygonic  $\mathrm{THH}$  version of Lemma 4.7.  $\square$

**Lemma 5.19.** *Let  $M$  be a  $\mathbb{Z}$ -module spectrum of bounded Tor-amplitude. Then the assembly map*

$$\mathrm{THH}(\mathbb{Z}; M^{\otimes_{\mathbb{Z}}^{\mathbb{I}} p})^{\theta_{\mathrm{THH}(\mathbb{Z})} C_p} \longrightarrow \mathrm{THH}(\mathbb{Z}; M^{\otimes_{\mathbb{Z}}^{\mathbb{I}} p})^{tC_p}$$

*is an equivalence after  $p$ -completion.*

**Proof.** Again, since both sides are exact in  $M$ , we may assume that  $M$  is a flat  $\mathbb{Z}$ -module, and the result is true for finite free  $\mathbb{Z}$ -modules  $M$ . By Lazard's theorem, the flat  $\mathbb{Z}$ -module  $M$  is a filtered colimit of finite free modules, thus it suffices to show that the functor

$$\begin{aligned} \mathrm{Mod}_{\mathbb{Z}}^b &\longrightarrow \mathrm{Sp}_p^\wedge \\ M &\longmapsto (\mathrm{THH}(\mathbb{Z}; M^{\otimes_{\mathbb{Z}}^{\mathbb{I}} p})^{tC_p})_p^\wedge \end{aligned}$$

preserves filtered colimits, where  $\mathrm{Mod}_{\mathbb{Z}}^b$  is the category of flat  $\mathbb{Z}$ -modules. Note that  $\mathrm{THH}(\mathbb{Z}; M^{\otimes_{\mathbb{Z}}^{\mathbb{I}} p})^{tC_p}$  acquires a  $\mathrm{THH}(\mathbb{Z})^{tC_p}$ -module structure, and in particular, a  $\mathbb{Z}$ -module structure, thus  $v_1 = 0$  on it. Now by Lemma 5.13, we have

$$\mathrm{THH}(\mathbb{Z}; M^{\otimes_{\mathbb{Z}}^{\mathbb{I}} p})^{tC_p} / (p, v_1) \simeq (\mathrm{THH}(\mathbb{Z}; M^{\otimes_{\mathbb{Z}}^{\mathbb{I}} p}) / (p, \tilde{\mu}_1))^{tC_p}$$

and as (non-equivariant) spectra, we have

$$\mathrm{THH}(\mathbb{Z}; M^{\otimes_{\mathbb{Z}}^{\mathbb{I}} p}) / (p, \tilde{\mu}_1) \simeq (\mathrm{THH}(\mathbb{Z}) / (p, \tilde{\mu}_1)) \otimes_{\mathbb{Z}}^{\mathbb{I}} M^{\otimes_{\mathbb{Z}}^{\mathbb{I}} p}$$

which is bounded, and preserves filtered colimits in  $M$ . The result follows from the fact that  $(-)^{tC_p}$  preserves uniformly bounded filtered colimits.  $\square$

Finally, we relate de-completed homotopy fixed points and de-completed Tate construction to de-completed Borel completion when the base is  $\mathrm{THH}(\mathbb{Z})^h$ . We would thank Georg TAMME for such a question. The point is that, as in the proofs of Proposition 5.15 and Lemma 5.18, after  $(p, v_1)$ -completion, the base  $\mathrm{THH}(\mathbb{Z})^h$  is close to be  $t$ -bounded. We start with some simple lemmas.

**Lemma 5.20.** *Let  $V \in \mathrm{Sp}^\omega$  be a finite spectrum, and  $A$  an  $\mathbb{E}_1$ -ring such that  $A \otimes V = 0$ . Then for every right  $A$ -module  $M$ , we have  $M \otimes V = 0$ .*

**Proof.** Note that  $M$  is a retract of  $M \otimes A$  as a spectrum, thus  $M \otimes V$  is a retract of  $M \otimes A \otimes V = 0$  as a spectrum.  $\square$

**Remark 5.21.** In our applications, we mainly take  $V$  to be the Smith–Toda complex  $\mathbb{S}/(p, v_1)$ .

**Lemma 5.22.** *Let  $V \in \mathrm{Sp}^\omega$  be a finite spectrum, and  $A_0 \rightarrow A$  a map of  $\mathbb{E}_1$ -algebras in  $\mathrm{Sp}^{B\mathbb{T}}$  such that  $A_0 \otimes V$  is  $t$ -bounded below. Suppose that the spectrum  $A_0^{tC_p} \otimes V$ , which is in fact equivalent to  $(A_0 \otimes V)^{tC_p}$ , is contractible. Then*

1. *for every right  $A$ -module  $M$ , the spectrum  $(M^{t\mathbb{T}})_p^\wedge \otimes V$  is contractible; and*
2. *the functor*

$$((-)^{h\mathbb{T}})_p^\wedge \otimes V : \mathrm{RMod}_A(\mathrm{Sp}^{B\mathbb{T}}) \longrightarrow \mathrm{Sp}_p^\wedge$$

*preserves filtered colimits.*

**Proof.** For the contractibility of  $(M^{t\mathbb{T}})_p^\wedge \otimes V$ , or equivalently, that of  $M^{t\mathbb{T}} \otimes (V/p)$ , by Lemma 5.20 and the lax symmetric monoidal structure on  $(-)^{t\mathbb{T}}$ , it suffices to show the contractibility of  $(A_0^{t\mathbb{T}})_p^\wedge \otimes V$ . But then

$$\begin{aligned} (A_0^{t\mathbb{T}})_p^\wedge \otimes V &\simeq ((A_0 \otimes V)^{t\mathbb{T}})_p^\wedge \\ &\simeq ((A_0 \otimes V)^{tC_p})^{h(\mathbb{T}/C_p)} \\ &\simeq 0, \end{aligned}$$

where the second equivalence follows from the  $t$ -bounded-belowness of  $A_0 \otimes V$  and [NS18, Lem II.4.2].

The filtered-colimit-preservation of the functor  $((-)^{h\mathbb{T}})_p^\wedge \otimes V$  follows directly from the canonical fiber sequence

$$\Sigma(-)_{h\mathbb{T}} \xrightarrow{\mathrm{Nm}_{\mathbb{T}}} (-)^{h\mathbb{T}} \longrightarrow (-)^{t\mathbb{T}}$$

and the Tate vanishing established above.  $\square$

**Proposition 5.23.** *Let  $A_0 \rightarrow A$  be a map of  $\mathbb{E}_1$ -algebras in  $\mathrm{Sp}^{B\mathbb{T}}$ , and  $V \in \mathrm{Sp}^\omega$  a finite spectrum such that  $A_0 \otimes V$  is  $t$ -bounded. Then*

1. *every  $A^{C_p}$ -module in  $\mathrm{Sp}^{g^<(\mathbb{T}/C_p)}$  is Borel-complete up to Bousfield  $V$ -localization; and*
2. *the  $p$ -completed de-completed homotopy fixed points functor*

$$((-)^{\eta_A \mathbb{T}})_p^\wedge : \mathrm{RMod}_A(\mathrm{Sp}^{g^<\mathbb{T}}) \longrightarrow \mathrm{Sp}_p^\wedge$$

*coincides with the  $p$ -completed genuine  $C_{p^\infty}$ -fixed points functor up to Bousfield  $V$ -localization, and the  $p$ -completed de-completed Tate construction functor*

$$((-)^{\theta_A \mathbb{T}})_p^\wedge : \mathrm{RMod}_A(\mathrm{Sp}^{g^<\mathbb{T}}) \longrightarrow \mathrm{Sp}_p^\wedge$$

*coincides with the composite functor*

$$\mathrm{RMod}_A(\mathrm{Sp}^{g^<\mathbb{T}}) \xrightarrow{(-)^{\Phi C_p}} \mathrm{RMod}_{A^{\Phi C_p}}(\mathrm{Sp}^{g^<(\mathbb{T}/C_p)}) \xrightarrow{(-)^{C_{p^\infty}/C_p}} \mathrm{Sp}_p^\wedge \quad (5.1)$$

*up to Bousfield  $V$ -localization.*

**Proof.** Firstly, for every  $m \in \mathbb{N}_{>1}$ , by Lemma 5.14, we have  $(A_0^{C_p})^{\Phi C_m} \otimes V \simeq (A_0 \otimes V)^{\Phi C_{mp}} \simeq 0$ . Then it follows from Lemma 5.20 and the lax symmetric monoidal structure on  $(-)^{\Phi C_p}$  and  $(-)^{C_m}$  that

1.  $(A^{C_p})^{\tau C_m} \otimes V = 0$ , and thus
2. for every right  $A^{C_p}$ -module  $M$  in  $\mathrm{Sp}^{g^<(\mathbb{T}/C_p)}$ , we have  $M^{\tau C_m} \otimes V = M^{\Phi C_m} \otimes V = 0$ .

In particular, every right  $A^{C_p}$ -module in  $\mathrm{Sp}^{g^<(\mathbb{T}/C_p)}$  is Borel-complete up to Bousfield  $V$ -localization.

By construction, on compact objects, we have the desired results. It suffices to show that the functors  $(-)^{C_{p^\infty}}$  and  $((-)^{\Phi C_p})^{C_{p^\infty}/C_p}$  preserve filtered colimits.

Note that  $(-)^{C_{p^\infty}} = ((-)^{C_p})^{C_{p^\infty}/C_p}$ , that is to say, we may write both functors  $(-)^{C_{p^\infty}}$  and  $((-)^{\Phi C_p})^{C_{p^\infty}/C_p}$  as a composite

$$\mathrm{RMod}_A(\mathrm{Sp}^{g^<\mathbb{T}}) \longrightarrow \mathrm{RMod}_{A^{C_p}}(\mathrm{Sp}^{g^<(\mathbb{T}/C_p)}) \xrightarrow{(-)^{C_{p^\infty}/C_p}} \mathrm{Sp}_p^\wedge,$$

where we are tacitly using the lax symmetric monoidal structure on the natural transformation  $(-)^{C_p} \Rightarrow (-)^{\Phi C_p}$ . Note that the first functor  $(-)^{\Phi C_p}$  or  $(-)^{C_p}$  preserves filtered colimits, thus it suffices to see that the second functor  $(-)^{C_{p^\infty}/C_p}$  preserves filtered colimits up to Bousfield  $V$ -localization.

We are now reduced to show that the functor

$$\mathrm{RMod}_{A^{hC_p}}(\mathrm{Sp}^{B(\mathbb{T}/C_p)}) \xrightarrow{(-)^{h(\mathbb{T}/C_p)} \otimes V} \mathrm{Sp}_p^\wedge$$

preserves filtered colimits. By examining the fiber sequence

$$\Sigma(-)_{h(\mathbb{T}/C_p)} \xrightarrow{\mathrm{Nm}_{\mathbb{T}/C_p}} (-)^{h(\mathbb{T}/C_p)} \longrightarrow (-)^{t(\mathbb{T}/C_p)}$$

It remains to show that, for every right  $A^{hC_p}$ -module  $M$  in  $\mathrm{Sp}^{B(\mathbb{T}/C_p)}$ , we have  $(M^{t(\mathbb{T}/C_p)})_p^\wedge \otimes V = 0$ . Actually, for every right  $A_0^{hC_p}$ -module  $M$ , the lax symmetric monoidal structure on  $(-)^{t(\mathbb{T}/C_p)}$  gives rise to a  $((A_0^{hC_p})^{t(\mathbb{T}/C_p)})_p^\wedge$ -module structure on  $(M^{t(\mathbb{T}/C_p)})_p^\wedge$ , and since  $A_0 \otimes V$  is  $t$ -bounded, by Lemma 5.24, we have

$$\begin{aligned} ((A_0^{hC_p})^{t(\mathbb{T}/C_p)})_p^\wedge \otimes V &\simeq (((A_0 \otimes V)^{hC_p})^{t(\mathbb{T}/C_p)})_p^\wedge \\ &\simeq 0 \end{aligned}$$

and thus  $(M^{t(\mathbb{T}/C_p)})_p^\wedge \otimes V = 0$  by Lemma 5.20.  $\square$

We need the following consequence of the Tate fixed point lemma.

**Lemma 5.24.** *Let  $M$  be a  $t$ -bounded  $\mathbb{T}$ -equivariant spectrum. Then  $((M^{hC_p})^{t(\mathbb{T}/C_p)})_p^\wedge = 0$ .*

**Proof.** By dévissage, we reduce to the case that  $M$  is concentrated in degree 0. In this case, we have

$$\begin{aligned} ((M^{hC_p})^{t(\mathbb{T}/C_p)})/p &\simeq (M^{hC_p})^{t(C_{p^2}/C_p)} \\ &\simeq 0, \end{aligned}$$

by the Tate fixed point lemma and [NS18, Lem IV.4.12].  $\square$

**Example 5.25.** Let  $k$  be an animated ring, and  $\mathcal{C}$  a presentable stable  $k$ -linear  $\infty$ -category. Then by Proposition 5.23, we have

$$\begin{aligned} \mathrm{HC}^{-, \mathrm{poly}}(\mathcal{C}/k)_p^\wedge &= ((\mathrm{HH}(\mathcal{C}/\underline{k})^{\eta_k})^{C_{p^\infty}})_p^\wedge, \\ \mathrm{HP}^{\mathrm{poly}}(\mathcal{C}/k)_p^\wedge &= (((\mathrm{HH}(\mathcal{C}/\underline{k})^{\eta_k})^{\Phi C_p})^{h(C_{p^\infty}/C_p)})_p^\wedge \\ &= ((\mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(k)} k^{tC_p})^{h(\mathbb{T}/C_p)})_p^\wedge. \end{aligned}$$

In retrospect, the object  $(\mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(k)} k^{tC_p})^{h(\mathbb{T}/C_p)}$  appeared [Dev23, Rem 3.5.6] in the even more general setup of  $k$  being an  $\mathbb{E}_\infty$ -ring.



Now we deduce the comparison over  $\mathrm{THH}(R)^h$  for every  $\mathbb{E}_\infty$ - $\mathbb{Z}$ -algebra  $R$ .

**Corollary 5.26.** *Let  $A$  be an  $\mathbb{E}_1$ - $\mathrm{THH}(\mathbb{Z})$ -algebra in  $\mathrm{Sp}^{B\mathbb{T}}$ . Then the  $(p, v_1)$ -completed de-completed homotopy fixed points functor*

$$((-)^{\eta_A \mathbb{T}})_{(p, v_1)}^\wedge : \mathrm{RMod}_A(\mathrm{Sp}^{g < \mathbb{T}}) \longrightarrow \mathrm{Sp}_{(p, v_1)}^\wedge$$

*coincides with the  $(p, v_1)$ -completed genuine  $C_{p^\infty}$ -fixed points functor, and the  $(p, v_1)$ -completed de-completed Tate construction functor*

$$((-)^{\theta_A \mathbb{T}})_{(p, v_1)}^\wedge : \mathrm{Mod}_A(\mathrm{Sp}^{g < \mathbb{T}}) \longrightarrow \mathrm{Sp}_{(p, v_1)}^\wedge$$

*coincides with the composite functor*

$$\mathrm{Mod}_A(\mathrm{Sp}^{g < \mathbb{T}}) \xrightarrow{(-)^{\Phi_{C_p}}} \mathrm{Mod}_{A^{\Phi_{C_p}}}(\mathrm{Sp}^{g < (\mathbb{T}/C_p)}) \xrightarrow{(-)^{C_{p^\infty}/C_p}} \mathrm{Sp}_{(p, v_1)}^\wedge.$$

**Proof.** As a very special case of [HW22, Thm G], the spectrum  $\mathrm{TR}(\mathbb{Z})/(p, v_1)$  is  $t$ -bounded. The result then follows from Proposition 5.23 by taking  $V = \mathbb{S}/(p, v_1)$ , and  $A_0 = \mathrm{TR}(\mathbb{Z})^h$ .  $\square$

**Remark 5.27.** In Corollary 5.26, we can replace  $\mathbb{Z}$  by  $\mathrm{BP}\langle n \rangle$ , and  $(p, v_1, \dots, v_{n+1})$  in place of  $(p, v_1)$ , for which we need the full [HW22, Thm G].

**Remark 5.28.** Let  $S$  be a perfectoid ring. Then the image of  $\mu_1 \in \pi_{2p}(\mathrm{THH}(\mathbb{Z})/p)$  under the map  $\mathrm{THH}(\mathbb{Z}) \rightarrow \mathrm{THH}(S)$  is  $\sigma^p$  up to a multiplier of a unit, where  $\sigma \in \pi_2(\mathrm{THH}(S)_p^\wedge)$  is the Bökstedt element. To see this, when  $S = \mathbb{F}_p$ , this is established in [BM94], which implies the case for  $S$  being a perfect  $\mathbb{F}_p$ -algebra, since the map  $\mathrm{THH}(\mathbb{Z}) \rightarrow \mathrm{THH}(S)$  factors through  $\mathrm{THH}(\mathbb{F}_p)$ . The general case follows by considering the composite

$$\mathrm{THH}(\mathbb{Z}) \longrightarrow \mathrm{THH}(S) \longrightarrow \mathrm{THH}(S/\sqrt{p}S)$$

and note that  $S/\sqrt{p}S$  is a perfect  $\mathbb{F}_p$ -algebra.

**Example 5.29.** Let  $S$  be a perfectoid ring, and  $\mathcal{C}$  a dualizable presentable stable  $S$ -linear  $\infty$ -category. Then we have

$$\begin{aligned} \mathrm{TC}^{-, \mathrm{poly}/S}(\mathcal{C}/S)_{(p, \xi)}^\wedge &= ((\mathrm{THH}(\mathcal{C})^{\eta_{\mathrm{THH}(S)}})^{C_{p^\infty}})_{(p, \xi)}^\wedge, \\ \mathrm{TP}^{\mathrm{poly}/S}(\mathcal{C}/S)_{(p, \xi)}^\wedge &= (((\mathrm{THH}(\mathcal{C})^{\eta_{\mathrm{THH}(S)}})^{\Phi_{C_p}})^{C_{p^\infty}/C_p})_{(p, \xi)}^\wedge. \end{aligned}$$

Indeed, they follow from Corollary 5.26 and Remark 5.28 by taking  $A := \mathrm{THH}(S)$  (where  $a_\lambda \tilde{\sigma} = \xi$  for the generator  $\tilde{\sigma} \in \pi_2(\mathrm{TC}^-(S)_p^\wedge)$  up to a complex orientation, and  $a_\lambda \tilde{\mu}_1 = v_1$ ).

Finally, we show that de-completed Borel completion over  $\mathrm{THH}(S)$  for perfectoid rings  $S$ , and over  $\mathbb{Z}$ , are “the same” as over  $\mathrm{THH}(\mathbb{Z})$ .

**Proposition 5.30.** *The assembly map  $\mathrm{THH}(S)^{\eta_{\mathrm{THH}(\mathbb{Z})}} \rightarrow \mathrm{THH}(S)^h$  in  $\mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})^h}(\mathrm{Sp}^{g < \mathbb{T}})$  is an equivalence after  $p$ -completion.*

**Proof.** By Corollary 5.17 and Lemmas 2.10 and 5.6, it is equivalent to show that the map

$$\mathrm{THH}(S) \otimes_{\mathrm{THH}(\mathbb{Z})}^{\mathbb{L}} \mathrm{THH}(\mathbb{Z})^{tC_p} \longrightarrow \mathrm{THH}(S)^{tC_p}$$

induced by the cyclotomic Frobenius  $\mathrm{THH}(S) \rightarrow \mathrm{THH}(S)^{tC_p}$  is an equivalence after modulo  $p$ , but this follows immediately from Remark 5.28.  $\square$

It then follows from Lemma 5.6 that

**Corollary 5.31.** *Let  $M$  be a  $\mathrm{THH}(S)$ -module in  $\mathrm{Sp}^{g < \mathbb{T}}$ . Then the assembly map*

$$M^{\eta_{\mathrm{THH}(\mathbb{Z})}} \longrightarrow M^{\eta_{\mathrm{THH}(S)}}$$

*in  $\mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})^h}(\mathrm{Sp}^{g < \mathbb{T}})$  is an equivalence after  $p$ -completion.*

**Remark 5.32.** More generally, we may replace  $\mathbb{Z}$  by any truncated Brown–Peterson spectrum. Indeed, for  $-1 \leq m < n$ , the assembly map

$$\mathrm{THH}(\mathrm{BP}\langle n \rangle)^{\eta_{\mathrm{THH}(\mathrm{BP}\langle m \rangle)}} \longrightarrow \mathrm{THH}(\mathrm{BP}\langle n \rangle)^h$$

in  $\mathrm{RMod}_{\mathrm{THH}(\mathrm{BP}\langle m \rangle)^h}(\mathrm{Sp}^{g^{< \mathbb{T}}})$  is an equivalence after  $(p, v_1, \dots, v_n)$ -completion, as a consequence of Remark 5.12. In particular, this shows that  $\mathrm{THH}_\Delta$  extends to a localizing invariant

$$\mathrm{Cat}_{\mathrm{BP}\langle n \rangle}^{\mathrm{perf}} \xrightarrow{\mathrm{THH}} \mathrm{RMod}_{\mathrm{THH}(\mathrm{BP}\langle n \rangle)}(\mathrm{Sp}^{g^{< \mathbb{T}}}) \xrightarrow{(-)^{\eta_{\mathrm{THH}(\mathrm{BP}\langle n \rangle)}}} \mathrm{RMod}_{\mathrm{THH}(\mathrm{BP}\langle n \rangle)^h}(\mathrm{Sp}^{g^{< \mathbb{T}}})_{(p, v_1, \dots, v_n)}^\wedge$$

which is independent of choice of  $n \in \mathbb{Z}_{\geq -1}$  in the sense of compatibility with forgetful functors.

On the other hand, we can also deduce the result for  $\mathbb{Z}$  from the case  $S = \mathbb{F}_p$ :

**Proposition 5.33.** *The assembly map  $\underline{\mathbb{Z}}^{\eta_{\mathrm{THH}(\mathbb{Z})}} \longrightarrow \mathbb{Z}$  in  $\mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})^h}(\mathrm{Sp}^{g^{< \mathbb{T}}})$  is an equivalence.*

**Proof.** Since  $\underline{\mathbb{Q}} \in \mathrm{CAlg}(\mathrm{Sp}^{g^{< \mathbb{T}}})$  is Borel, it suffices to show this equivalence after modulo every prime  $p$ . By Corollary 5.17 and Lemmas 2.10 and 5.6, we are reduced to show that the map

$$\tau_{\geq 0}(\mathbb{F}_p^{tC_p}) \otimes_{\mathrm{THH}(\mathbb{Z})}^{\mathbb{L}} \mathrm{THH}(\mathbb{Z})^{tC_p} \longrightarrow \mathbb{F}_p^{tC_p}$$

is an equivalence, where the  $\mathbb{E}_\infty$ -map  $\mathrm{THH}(\mathbb{Z}) \rightarrow \tau_{\geq 0}(\mathbb{F}_p^{tC_p})$  is the composite

$$\mathrm{THH}(\mathbb{Z}) \xrightarrow{\sim} \tau_{\geq 0}(\mathrm{THH}(\mathbb{Z})^{tC_p}) \longrightarrow \tau_{\geq 0}(\mathbb{Z}^{tC_p}) \longrightarrow \tau_{\geq 0}(\mathbb{F}_p^{tC_p})$$

where the first equivalence is the Segal conjecture for  $\mathrm{THH}(\mathbb{Z})$ . Now this composite can be rewritten as the composite

$$\mathrm{THH}(\mathbb{Z}) \xrightarrow{\sim} \tau_{\geq 0}(\mathrm{THH}(\mathbb{Z})^{tC_p}) \longrightarrow \tau_{\geq 0}(\mathrm{THH}(\mathbb{F}_p)^{tC_p}) \longrightarrow \tau_{\geq 0}(\mathbb{F}_p^{tC_p}),$$

and we have a composite pushout diagram

$$\begin{array}{ccccc} \tau_{\geq 0}(\mathrm{THH}(\mathbb{Z})^{tC_p}) & \longrightarrow & \tau_{\geq 0}(\mathrm{THH}(\mathbb{F}_p)^{tC_p}) & \longrightarrow & \tau_{\geq 0}(\mathbb{F}_p^{tC_p}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{THH}(\mathbb{Z})^{tC_p} & \longrightarrow & \mathrm{THH}(\mathbb{F}_p)^{tC_p} & \longrightarrow & \mathbb{F}_p^{tC_p} \end{array}$$

of  $\mathbb{E}_\infty$ -rings, where the left pushout is Proposition 5.30, and the right pushout is simply detected by both horizontal maps are inverting the Bökstedt element  $\mu_0 \in \pi_2(\mathrm{THH}(\mathbb{F}_p)^{tC_p})$ .  $\square$

**Remark 5.34.** Let  $p$  be an odd prime. [DR25, Prop 5.2.1] implies that the map  $\mathrm{THH}(\mathbb{Z})^{tC_p} \rightarrow \mathbb{Z}^{tC_p}$  coincides with the map  $\mathrm{THH}(\mathbb{Z})^{tC_p} \rightarrow \mathrm{THH}(\mathbb{F}_p)^{tC_p}$  induced by the ring map  $\mathbb{Z} \rightarrow \mathbb{F}_p$ .

**Question 4.** Is the statement of Remark 5.34 true when  $p = 2$ ?

**Corollary 5.35.** *Let  $M$  be a  $\underline{\mathbb{Z}}$ -module in  $\mathrm{Sp}^{g^{< \mathbb{T}}}$ . Then the assembly map*

$$M^{\eta_{\mathrm{THH}(\mathbb{Z})}} \longrightarrow M^{\eta_{\underline{\mathbb{Z}}}}$$

*in  $\mathrm{Mod}_{\mathrm{THH}(\mathbb{Z})^h}(\mathrm{Sp}^{g^{< \mathbb{T}}})$  is an equivalence.*

## 6 Noncommutative crystalline--de Rham comparison revisited

In this section, we explain that original ideas in [PV19], along with an observation by A. RAKSIT, adapt to de-completed case, which leads to a comparison (Corollary 6.28), which “authentically” corresponds to the crystalline–de Rham comparison. As explained in Remark 6.27, the argument in [DR25] is enough for this comparison when  $p$  is an odd prime, so our result is stronger only when  $p = 2$ .

The key is to produce a map  $\mathrm{HH}(\mathbb{F}_p/\mathbb{Z})^{\theta_{\mathbb{Z}} C_p} \rightarrow \mathbb{Z}^{t C_p}$  (Construction 6.15). As A. RAKSIT observed, up to  $(-)^{t C_p}$ , we can replace  $\mathrm{HH}(\mathbb{Z}[t]/\mathbb{Z})$  by  $\mathrm{dR}_{\mathbb{Z}[t]/\mathbb{Z}}$ , and consequently, we can identify  $\mathrm{HH}(\mathbb{F}_p/\mathbb{Z})^{\theta_{\mathbb{Z}} C_p}$  with  $\mathrm{dR}_{\mathbb{F}_p/\mathbb{Z}}^{t C_p}$ . Combining with the map  $\mathrm{dR}_{\mathbb{F}_p/\mathbb{Z}} \rightarrow \mathbb{Z}_{(p)}$  induced by the PD-structure on  $(p) \subseteq \mathbb{Z}_{(p)}$ , we get the map we want. As a consequence, we get a map  $\mathrm{THH}(\mathbb{F}_p)^{t C_p} \rightarrow \mathrm{HH}(\mathbb{F}_p/\mathbb{Z})^{\theta_{\mathbb{Z}} C_p} \rightarrow \mathbb{Z}^{t C_p}$ . We then show that this is an equivalence (Proposition 6.25), the blueprint of whose proof can be found after Construction 6.16.

We start with A. RAKSIT's observation. Let  $k$  be a commutative ring, and  $R$  a smooth commutative  $k$ -algebra. In general, we are only equipped with an HKR-filtration on the periodic cyclic homology whose associated graded pieces are equivalent to shifts of Hodge-completed derived de Rham cohomology of  $R/k$ . However, when  $R$  is of relative dimension  $\leq 1$ , the filtration in question, along with the commutative ring structure<sup>6.1</sup>, is simply extended from the filtration on  $k^{t\mathbb{T}}$ , which was observed by A. Raksit as in Construction 6.13. A slight improvement of his argument gives a stronger result: we compare associated  $[0, 1]$ -graded pieces of HKR-filtered Hochschild homology and Hodge-filtered de Rham cohomology in Corollary 6.12. For this purpose, we first review some facts on homotopy coherent cochain complexes with weights in  $[0, 1]$ , which is systematically studied in K. MAGIDSON's work [Mag24].

**Notation 6.1.** Let  $k$  be a commutative ring. We denote by  $\mathrm{DG\_DAlg}_k$  the  $\infty$ -category of  $h_-$ -differential graded derived commutative  $k$ -algebras [Rak20, Nota 5.3.1], and  $\mathrm{DG}_{-}^{[0,1]} \mathrm{DAlg}_k \subseteq \mathrm{DG\_DAlg}_k$  the full subcategory of objects whose weights are concentrated in degrees  $[0, 1]$ .

**Remark 6.2.** ([Mag24, §3.3]) Let  $k$  be a commutative ring. Then the  $\infty$ -category  $\mathrm{DG}_{-}^{[0,1]} \mathrm{DAlg}_k$  is equivalent to the  $\infty$ -category of triples  $(A, M, \eta)$  where

- $A$  is a derived commutative  $k$ -algebra;
- $M$  is an  $A$ -module; and
- $\eta: A \rightarrow M$  is a derivation (or equivalently, a map  $L_{A/k} \rightarrow M$  of  $A$ -modules).

Given a triple  $(A, M, \eta)$  as above, the corresponding object in  $\mathrm{DG}_{-}^{[0,1]} \mathrm{DAlg}_k$  is equivalent to the square-zero extension of  $A(0)$  by  $M(1)[-1]$  determined by the derivation  $\eta$ , whose underlying  $k$ -module spectrum is equivalent to  $\mathrm{fib}(A \rightarrow M)$ .

The key construction is the following:

**Construction 6.3.** Let  $k$  be a commutative ring. We construct a functor

$$\mathrm{DG}_{-}^{[0,1]} \mathrm{DAlg}_k \longrightarrow \mathrm{DAlg}(D(k)^{B\mathbb{T}})$$

as follows. Let  $(A, M, \eta)$  be a triple corresponding to an object of  $\mathrm{DG}_{-}^{[0,1]} \mathrm{DAlg}_k$  as in Remark 6.2. Equip  $A$  and  $M$  with trivial  $\mathbb{T}$ -action, we get a derivation  $\eta: A \rightarrow M$  in  $\mathrm{DAlg}(D(k)^{B\mathbb{T}})$ . Now we compose this derivation with the map  $M = M \otimes \mathbb{S} \xrightarrow{M \otimes a_\lambda} M \otimes \mathbb{S}^\lambda = \Sigma^\lambda M$ , obtaining another derivation  $\tau: A \rightarrow \Sigma^\lambda M$  in  $\mathrm{DAlg}(D(k)^{B\mathbb{T}})$ . We take the square-zero extension  $[A \xrightarrow{\tau} \Sigma^\lambda M]$  of  $A$  by  $\Sigma^\lambda M[-1]$  determined by the derivation  $\tau$ . This gives rise to a functor

$$\begin{aligned} \mathrm{DG}_{-}^{[0,1]} \mathrm{DAlg}_k &\longrightarrow \mathrm{DAlg}(D(k)^{B\mathbb{T}}) \\ (A, M, \eta) &\longmapsto [A \xrightarrow{\tau} \Sigma^\lambda M]. \end{aligned}$$

Moreover, the map  $M \otimes a_\lambda: M \rightarrow \Sigma^\lambda M$  induces a map  $[A \xrightarrow{\eta} M] \rightarrow [A \xrightarrow{\tau} \Sigma^\lambda M]$  of derived commutative algebras in  $D(k)^{B\mathbb{T}}$ , where  $[A \xrightarrow{\eta} M]$  is equipped with trivial  $\mathbb{T}$ -action.

**Remark 6.4.** In fact, in Construction 6.3, we can endow the target a filtered circle action. Since this notion is not used in the text, we will not discuss this filtered enhancement.

We need the following categorification of Tate construction.

---

6.1. The key is the commutative ring structure.

**Definition 6.5.** ([PV19, §2]) Let  $k$  be a commutative ring spectrum. The  $\mathbb{T}$ -Tate category  $D(k)^{t\mathbb{T}}$  is the quotient category

$$D(k)^{B\mathbb{T}} / \langle M \otimes \mathbb{T} \mid M \in D(k) \rangle$$

where  $\langle M \otimes \mathbb{T} \mid M \in D(k) \rangle \subseteq D(k)^{B\mathbb{T}}$  is the stable subcategory generated by induced  $\mathbb{T}$ -representations.

The magic of Construction 6.3 is that it does not change the Tate construction, which is formally formulated as follows.

**Proposition 6.6.** Let  $k$  be a commutative ring, and  $(A, M, \eta)$  a triple corresponding to an object of  $\mathrm{DG}_{-}^{[0,1]} \mathrm{DAlg}_k$ . Then the map

$$[A \xrightarrow{\eta} M] \longrightarrow [A \xrightarrow{\tau} \Sigma^\lambda M]$$

in  $\mathrm{DAlg}(D(k)^{B\mathbb{T}})$  constructed in Construction 6.3, where  $[A \xrightarrow{\eta} M]$  is equipped with trivial  $\mathbb{T}$ -action, is an equivalence after passing along the quotient functor  $D(k)^{B\mathbb{T}} \rightarrow D(k)^{t\mathbb{T}}$ .

**Proof.** The point is that  $\mathrm{fib}(a_\lambda : \mathbb{S} \rightarrow \mathbb{S}^\lambda) \simeq \mathbb{S} \otimes \mathbb{T}$  is an induced  $\mathbb{T}$ -representation.  $\square$

Let  $k \rightarrow R$  be a smooth map of commutative algebras. The next goal, Corollary 6.12, is to show that, if we apply this construction to the truncated de Rham complex

$$0 \longrightarrow R \xrightarrow{d} \Omega_{R/k}^1 \longrightarrow 0$$

we get the associated  $[0, 1]$ -graded piece of the Hochschild homology  $\mathrm{HH}(R/k)$ . We explain this more formally for animated  $k$ -algebras  $R$  as follows.

**Remark 6.7.** Let  $k$  be a commutative ring. The forgetful functor  $D(k)^{B\mathbb{T}} = \mathrm{coAlg}_{k[\mathbb{T}]^\vee}(D(k)) \rightarrow D(k)$  maps derived commutative algebras in  $D(k)^{B\mathbb{T}}$  to derived commutative  $k$ -algebras, and it carries the adic filtration on a map  $f$  in  $\mathrm{DAlg}(D(k)^{B\mathbb{T}})$  to the adic filtration on the underlying map  $f$  in  $\mathrm{DAlg}_k$ .

**Lemma 6.8.** Let  $k$  be a commutative ring, and  $A$  an animated commutative  $k$ -algebra. Then the HKR-filtration on  $\mathrm{HH}(A/k)$ , as a filtered derived commutative algebra in  $D(k)^{B\mathbb{T}}$ , coincides with the adic filtration on the map  $\mathrm{HH}(A/k) \rightarrow A$  in  $\mathrm{DAlg}(D(k)^{B\mathbb{T}})$ .

**Proof.** By the universal property of adic filtration, we get a map

$$\mathrm{Fil}_{\mathrm{ad}}(\mathrm{HH}(A/k) \rightarrow A) \longrightarrow \mathrm{Fil}_{\mathrm{HKR}} \mathrm{HH}(A/k)$$

in  $\mathrm{Fil} \mathrm{DAlg}(D(k)^{B\mathbb{T}})$ . To show that this is an equivalence, by conservativity, it suffices to show the equivalence after passing along the forgetful functor  $\mathrm{Fil} \mathrm{DAlg}(D(k)^{B\mathbb{T}}) \rightarrow \mathrm{Fil} \mathrm{DAlg}_k$ , and then it follows from Remark 6.7 that the left hand side is still the adic filtration of the map  $\mathrm{HH}(A/k) \rightarrow A$ , and on the level of  $\mathrm{Fil} \mathrm{DAlg}_k$ , this equivalence is true for polynomial  $k$ -algebras  $A$ , thus also for animated  $k$ -algebras  $A$ .  $\square$

**Remark 6.9.** Let  $\mathcal{C}$  be  $D(k)$  or  $D(k)^{B\mathbb{T}}$ , and  $A \rightarrow B$  be a map of derived commutative algebras in  $\mathcal{C}$ . Then the derived commutative algebra  $\mathrm{gr}_{\mathrm{ad}}^{[0,1]}(A \rightarrow B)$  can be realized as the square-zero extension

$$\left[ B \xrightarrow{d} L_B \xrightarrow{\mathrm{can}} L_{B/A} \right]$$

of  $B$  by  $L_{B/A}[-1]$  in  $\mathcal{C}$ .

**Lemma 6.10.** Let  $k$  be a commutative ring, and  $A$  a derived commutative  $k$ -algebra. Then the canonical map

$$L_{\mathrm{HH}(A/k)/k} \otimes_{\mathrm{HH}(A/k)}^{\mathbb{L}} A \longrightarrow L_{A/k}$$

in  $\mathrm{Mod}_A(D(k)^{B\mathbb{T}})$  can be identified with the map

$$L_{A/k} \otimes_k^{\mathbb{L}} k[\mathbb{T}] \longrightarrow L_{A/k}$$

induced by the  $\mathbb{T}$ -equivariant projection  $\mathbb{T} \rightarrow *$ .

**Proof.** Let  $X \in \mathbf{An}$  be an anima, and  $B_X := A^{\otimes_{\mathbb{L}} X}$  the  $X$ -th tensor power of  $A$ . Then we have an equivalence

$$L_{B_X/k} \otimes_{B_X}^{\mathbb{L}} A \simeq L_{A/k} \otimes_k^{\mathbb{L}} k[X]$$

in  $\mathrm{Mod}_A(D(k))$ , which is functorial in  $X \in \mathbf{An}$ . Now we restrict this to the full subcategory  $(B\mathbb{T})^{\triangleright} \subseteq \mathbf{An}$  spanned by  $\{\mathbb{T}, *\}$ , we obtained the identification that we want.  $\square$

**Proposition 6.11.** *Let  $k$  be a commutative ring, and  $A$  an animated commutative  $k$ -algebra. Then the derived commutative algebra  $\mathrm{gr}_{\mathrm{HKR}}^{[0,1]} \mathrm{HH}(A/k)$  can be identified with the square-zero extension*

$$\left[ A \xrightarrow{d} L_{A/k} \xrightarrow{L_{A/k} \otimes a_\lambda} \Sigma^\lambda L_{A/k} \right]$$

of  $A$  by  $\Sigma^{\lambda-1} L_{A/k} \simeq L_{A/k}[1]$  (via a complex orientation of  $k$ ) in  $\mathrm{DAlg}(D(k)^{B\mathbb{T}})$ .

**Proof.** By Remark 6.9 and Lemma 6.8, it suffices to identify the map

$$f : L_{A/k} \longrightarrow L_{A/\mathrm{HH}(A/k)}$$

in  $\mathrm{Mod}_A(D(k)^{B\mathbb{T}})$  with the map

$$L_{A/k} \xrightarrow{L_{A/k} \otimes a_\lambda} \Sigma^\lambda L_{A/k}.$$

We examine the transitivity sequence

$$L_{\mathrm{HH}(A/k)/k} \otimes_{\mathrm{HH}(A/k)}^{\mathbb{L}} A \longrightarrow L_{A/k} \xrightarrow{f} L_{A/\mathrm{HH}(A/k)}$$

in  $\mathrm{Mod}_A(D(k)^{B\mathbb{T}})$  associated to the composite map

$$k \longrightarrow \mathrm{HH}(A/k) \longrightarrow A$$

in  $\mathrm{DAlg}(D(k)^{B\mathbb{T}})$ . By Lemma 6.10, the map  $f$  fits into a fiber sequence

$$L_{A/k} \otimes_k^{\mathbb{L}} k[\mathbb{T}] \longrightarrow L_{A/k} \xrightarrow{f} L_{A/\mathrm{HH}(A/k)}$$

in  $\mathrm{Mod}_A(D(k)^{B\mathbb{T}})$ , and the result follows from the fiber sequence

$$\mathbb{S} \xrightarrow{a_\lambda} \mathbb{S}^\lambda \longrightarrow \mathbb{S} \otimes \mathbb{T}. \quad \square$$

**Corollary 6.12.** *Let  $k$  be a commutative ring, and  $A$  an animated commutative  $k$ -algebra. Then the derived commutative algebra  $\mathrm{gr}_{\mathrm{HKR}}^{[0,1]} \mathrm{HH}(A/k) \in \mathrm{DAlg}(D(k)^{B\mathbb{T}})$  can be identified with the image of  $L\Omega_{A/k}^{[0,1]} \in \mathrm{DG}_-^{[0,1]} \mathrm{DAlg}_k$  under the functor in Construction 6.3.*

Now we restrict to smooth maps of dimension  $\leq 1$ . In this case, both HKR-filtration and Hodge-filtration are concentrated in weights  $[0, 1]$ , thus we have

**Construction 6.13. (Raksit)** Let  $k$  be a commutative ring, and  $A$  a smooth  $k$ -algebra of dimension  $\leq 1$ . Then by Construction 6.3 and Corollary 6.12, we get a map

$$\mathrm{dR}_{R/k} \longrightarrow \mathrm{HH}(R/k)$$

in  $\mathrm{DAlg}(D(k)^{B\mathbb{T}})$ , which becomes an equivalence in  $D(k)^{t\mathbb{T}}$  by Proposition 6.6. This construction is functorial in smooth maps  $k \rightarrow R$  of relative dimension  $\leq 1$ .

Now we want to apply this to analyze the de-completed  $C_p$ -Tate construction  $\mathrm{HH}(\mathbb{F}_p/\mathbb{Z})^{\theta_{\mathbb{Z}} C_p}$ .

**Remark 6.14.** The composite lax symmetric monoidal functor

$$D(\mathbb{Z})^{B\mathbb{T}} \xrightarrow{(-)^h} \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}^{g < \mathbb{T}}) \xrightarrow{(-)^{\theta_{\mathbb{Z}} C_p}} \mathrm{Mod}_{\mathbb{Z}^{tC_p}}(\mathrm{Sp}^{B(\mathbb{T}/C_p)})$$

factors canonically through  $D(\mathbb{Z})^{t\mathbb{T}}$ .

**Construction 6.15.** Applying Construction 6.13 to the coCartesian diagram

$$\begin{array}{ccc} k[t] & \xrightarrow{t \mapsto p} & k \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p \end{array},$$

of animated  $k$ -algebras, we get a map

$$\mathrm{dR}_{(k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p)/k} \longrightarrow \mathrm{HH}(k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p/k)^{\eta_k}$$

of derived commutative algebras in  $\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}^{g < \mathbb{T}})$ , which becomes an equivalence after taking  $(-)^{\Phi_{C_p}}$  by Remark 6.14. In particular, when  $k = \mathbb{Z}_{(p)}$ , the map

$$\mathrm{dR}_{\mathbb{F}_p/\mathbb{Z}_{(p)}} = D_{\mathbb{Z}_{(p)}}(p) \longrightarrow \mathbb{Z}_{(p)}$$

of augmented (derived) commutative  $\mathbb{Z}_{(p)}$ -algebras (where  $D_{\mathbb{Z}_{(p)}}(p)$  is the PD-envelope of  $(p) \subseteq \mathbb{Z}_{(p)}$ ), induced by the PD-structure on  $(\mathbb{Z}_{(p)}, (p))$ , gives rise to a map

$$\mathrm{HH}(\mathbb{F}_p/\mathbb{Z})^{\theta_{\mathbb{Z}} C_p} \rightarrow \mathbb{Z}^{t C_p}$$

of  $\mathbb{E}_{\infty}\text{-}\mathbb{Z}^{t C_p}$ -algebras in  $\mathrm{Sp}^{B(\mathbb{T}/C_p)}$ .

**Construction 6.16.** Composing with the map  $\mathrm{THH}(\mathbb{F}_p)^h \rightarrow \mathrm{HH}(\mathbb{F}_p/\mathbb{Z})^{\eta_{\mathbb{Z}}}$  of  $\mathbb{E}_{\infty}\text{-}\mathrm{THH}(\mathbb{Z})^h$ -algebras in  $\mathrm{Sp}^{g < \mathbb{T}}$ , we get a composite map

$$\mathrm{THH}(\mathbb{F}_p)^{t C_p} \longrightarrow \mathrm{HH}(\mathbb{F}_p/\mathbb{Z})^{\theta_{\mathbb{Z}} C_p} \longrightarrow \mathbb{Z}^{t C_p}$$

of  $\mathbb{E}_{\infty}\text{-}\mathrm{THH}(\mathbb{Z})^{t C_p}$ -algebras in  $\mathrm{Sp}^{B(\mathbb{T}/C_p)}$ .

**Remark 6.17.** The map in Construction 6.16 lives in the Borel equivariant world, while our construction relies on genuine equivariant homotopy theory.

Our next goal is to show that the map in Construction 6.16 is an equivalence. We check it after modulo  $p$ . The point is that, although we do not know whether this map on the nose acquires a  $\mathbb{Z}$ -linear structures, its modulo  $p$  has an  $\mathbb{F}_p$ -linear structure. More precisely, our argument follows the following strategy<sup>6.2</sup> in [PVV18]:

1. Give the composite map  $\mathrm{THH}(\mathbb{F}_p)^{t C_p} \rightarrow \mathbb{Z}^{t C_p} \rightarrow \mathbb{F}_p^{t C_p}$  a  $\mathbb{Z}$ -linear structure (Construction 6.20);
2. Show that, under this  $\mathbb{Z}$ -linear structure, the induced map  $\mathrm{THH}(\mathbb{F}_p)^{t C_p} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p \rightarrow \mathbb{F}_p^{t C_p}$  is an equivalence (Lemma 6.21);
3. Show that the induced map above coincides with the modulo  $p$  reduction of the map  $\mathrm{THH}(\mathbb{F}_p)^{t C_p} \rightarrow \mathbb{Z}^{t C_p}$  in Construction 6.16 (Lemma 6.24).

**Remark 6.18.** Concretely, the map  $D_{\mathbb{Z}_{(p)}}(p) \rightarrow \mathbb{Z}_{(p)}$  is given by the map

$$\begin{aligned} \mathbb{Z}_{(p)} \left[ t, \frac{t^2}{2!}, \frac{t^3}{3!}, \dots \right] / (t-p) &\longrightarrow \mathbb{Z}_{(p)} \\ \frac{t^n}{n!} &\longmapsto \frac{p^n}{n!}, \end{aligned}$$

and therefore, after (derived) modulo  $p$ , it becomes the augmentation map

$$\begin{aligned} \mathbb{F}_p \left[ t, \frac{t^p}{p}, \frac{t^{p^2}}{p^{p+1}}, \dots \right] / (t) &\longrightarrow \mathbb{F}_p \\ \frac{t^{p^r}}{p^{p^{r-1} + \dots + 1}} &\longmapsto 0. \end{aligned}$$

It follows that, the map  $\mathrm{dR}_{\mathbb{F}_p/\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}}^{\mathbb{L}} \mathbb{F}_p \rightarrow \mathbb{F}_p$  induced by the PD-structure on  $(p) \subseteq \mathbb{Z}_{(p)}$  coincides with the map  $\mathrm{dR}_{k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p/k} \rightarrow \mathrm{dR}_{k/k} = k$  induced by the multiplication map  $k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p \rightarrow k$ , for derived commutative  $\mathbb{F}_p$ -algebras  $k$ .

<sup>6.2</sup> We thank A. PETROV for explaining this.



We get a de-completed version of [PV19, Cor 2.7] (without restricting to odd primes):

**Corollary 6.19.** *The modulo  $p$  reduction of the map  $\mathrm{HH}(\mathbb{F}_p/\mathbb{Z})^{\Phi_{C_p}} \rightarrow \mathbb{Z}^{t_{C_p}}$  of  $\mathbb{E}_\infty$ - $\mathbb{Z}^{t_{C_p}}$ -algebras in  $\mathrm{Sp}^{B(\mathbb{T}/C_p)}$  in Construction 6.15 becomes the map  $\mathrm{HH}(k \otimes_{\mathbb{Z}} \mathbb{F}_p/k)^{\Phi_{C_p}} \rightarrow \mathrm{HH}(k/k)^{\Phi_{C_p}}$  of  $\mathbb{E}_\infty$ - $\mathbb{F}_p^{t_{C_p}}$ -algebras in  $\mathrm{Sp}^{B(\mathbb{T}/C_p)}$  induced by the multiplication map  $k \otimes_{\mathbb{Z}} \mathbb{F}_p \rightarrow k$ , for  $k = \mathbb{F}_p$ <sup>6.3</sup>.*

**Construction 6.20.** Let  $k = \mathbb{F}_p$ . The composite map in Construction 6.16 fits into a commutative diagram

$$\begin{array}{ccccc} \mathrm{THH}(k)^{t_{C_p}} & \longrightarrow & \mathrm{HH}(k/\mathbb{Z})^{\theta_{\mathbb{Z}} C_p} & \longrightarrow & \mathbb{Z}^{t_{C_p}} \\ & & \downarrow & & \downarrow \\ & & \mathrm{HH}(k \otimes_{\mathbb{Z}} \mathbb{F}_p/\mathbb{F}_p)^{\theta_{\mathbb{F}_p} C_p} & \longrightarrow & \mathbb{F}_p^{t_{C_p}} \end{array}$$

of  $\mathbb{E}_\infty$ - $\mathrm{THH}(\mathbb{Z})^{t_{C_p}}$ -algebras in  $\mathrm{Sp}^{B(\mathbb{T}/C_p)}$ . It follows from Corollary 6.19 that the composite map  $\mathrm{THH}(k)^{t_{C_p}} \rightarrow \mathbb{F}_p^{t_{C_p}}$  is equivalent to applying  $(-)^{t_{C_p}}$  to the map

$$\mathrm{THH}(\mathbb{F}_p) \longrightarrow \mathrm{HH}(\mathbb{F}_p/\mathbb{F}_p)$$

induced by the map  $(\mathbb{Z} \rightarrow \mathbb{F}_p) \rightarrow (\mathbb{F}_p \rightarrow \mathbb{F}_p)$  of maps of  $\mathbb{E}_\infty$ -rings, which acquires a structure of maps of  $\mathbb{E}_\infty$ - $\mathbb{Z}^{t_{C_p}}$ -algebras in  $\mathrm{Sp}^{B(\mathbb{T}/C_p)}$ . This map is also the same as the multiplication map  $\mathrm{THH}(\mathbb{F}_p) \rightarrow \mathbb{F}_p$ .

**Lemma 6.21.** *The map  $\mathrm{THH}(\mathbb{F}_p)^{t_{C_p}} \otimes_{\mathbb{Z}} \mathbb{F}_p \rightarrow \mathbb{F}_p^{t_{C_p}}$  induced by the  $\mathbb{Z}^{t_{C_p}}$ -linear map  $\mathrm{THH}(\mathbb{F}_p)^{t_{C_p}} \rightarrow \mathbb{F}_p^{t_{C_p}}$  in Construction 6.20 is an equivalence.*

**Proof.** The composite map  $\mathbb{Z} \rightarrow \mathrm{THH}(\mathbb{F}_p) \rightarrow \mathbb{F}_p$  is the canonical map since it is a map of commutative algebras in the heart of  $D(\mathbb{Z})^{B\mathbb{T}}$ . By [NS18, Cor IV.4.13], the first map becomes an equivalence after taking  $(-)^{t_{C_p}}$ , and the result follows.  $\square$

Finally, we are in the following slightly tricky situation: given two  $\mathbb{Z}$ -modules  $M$  and  $N$ , along with a (non- $\mathbb{Z}$ -linear) map  $f: M \rightarrow N$ . Suppose that the composite map  $g: M \rightarrow N \rightarrow N \otimes_{\mathbb{Z}} \mathbb{F}_p$  carries a  $\mathbb{Z}$ -linear structure, which induces a map  $M \otimes_{\mathbb{Z}} \mathbb{F}_p \rightarrow N \otimes_{\mathbb{Z}} \mathbb{F}_p$ . We claim that this map coincides with the modulo  $p$  reduction of  $f$ . We break this into two steps. The first is to produce the map  $M/p \rightarrow N \otimes_{\mathbb{Z}} \mathbb{F}_p$  without the  $\mathbb{Z}$ -linear structure on  $g$ .

**Construction 6.22.** Let  $\mathcal{C}$  be a presentable stable  $\infty$ -category, and  $M \in \mathcal{C}$  an object, and  $L \in \mathrm{Mod}_{\mathbb{F}_p}(\mathcal{C})$  an  $\mathbb{F}_p$ -module in  $\mathcal{C}$ , and  $f: M \rightarrow L$  a map in  $\mathcal{C}$ . This gives rise to a composite map

$$M/p \longrightarrow L/p \simeq L \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\mathrm{mult}} L.$$

**Lemma 6.23.** *Let  $\mathcal{C}$  be a presentable stable  $\infty$ -category, and  $M \in \mathrm{Mod}_{\mathbb{Z}}(\mathcal{C})$  a  $\mathbb{Z}$ -module in  $\mathcal{C}$ , and  $L \in \mathrm{Mod}_{\mathbb{F}_p}(\mathcal{C})$  an  $\mathbb{F}_p$ -module in  $\mathcal{C}$ , and  $f: M \rightarrow L$  a  $\mathbb{Z}$ -linear map in  $\mathcal{C}$ . Then the map  $M \otimes_{\mathbb{Z}} \mathbb{F}_p \rightarrow L$  in  $\mathrm{Mod}_{\mathbb{F}_p}(\mathcal{C})$ , as a map in  $\mathcal{C}$ , coincides with the map  $M/p \rightarrow L$  in Construction 6.22.*

**Proof.** The map  $f \otimes_{\mathbb{Z}} \mathbb{F}_p: M \otimes_{\mathbb{Z}} \mathbb{F}_p \rightarrow N \otimes_{\mathbb{Z}} \mathbb{F}_p$  in  $\mathrm{Mod}_{\mathbb{F}_p}(\mathcal{C})$ , as a map in  $\mathcal{C}$ , coincides with the map  $f/p: M/p \rightarrow N/p$ . The result follows.  $\square$

**Lemma 6.24.** *Let  $\mathcal{C}$  be a presentable stable  $\infty$ -category, with  $M$  and  $N$  two  $\mathbb{Z}$ -modules in  $\mathcal{C}$ , and  $f: M \rightarrow N$  a map in  $\mathcal{C}$ . We give the composite map  $g: M \xrightarrow{f} N \rightarrow N \otimes_{\mathbb{Z}} \mathbb{F}_p$  a  $\mathbb{Z}$ -linear structure, which induces a map  $h: M \otimes_{\mathbb{Z}} \mathbb{F}_p \rightarrow N \otimes_{\mathbb{Z}} \mathbb{F}_p$  in  $\mathrm{Mod}_{\mathbb{F}_p}(\mathcal{C})$ . Then as a map in  $\mathcal{C}$ , the map  $h$  coincides with the map  $f/p$ .*

**Proof.** By Lemma 6.23, the map  $h$ , as a map in  $\mathcal{C}$ , coincides with the composite

$$M/p \xrightarrow{g/p} (N \otimes_{\mathbb{Z}} \mathbb{F}_p)/p \simeq N \otimes_{\mathbb{Z}} \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\mathrm{id}_N \otimes \mathrm{mult}} N \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

<sup>6.3</sup> Although we only apply to the case that  $k = \mathbb{F}_p$ , this notation allows us to distinguish different  $\mathbb{F}_p$ -algebra structures on  $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p$ .

Thus it suffices to show that the composite

$$N/p \rightarrow (N \otimes_{\mathbb{Z}} \mathbb{F}_p)/p \simeq N \otimes_{\mathbb{Z}} \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\text{id}_N \otimes \text{mult}} N \otimes_{\mathbb{Z}} \mathbb{F}_p$$

is equivalent to identity. Note that the first map can be identified with

$$N \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\text{id}_N \otimes 1 \otimes \text{id}_{\mathbb{F}_p}} N \otimes_{\mathbb{Z}} \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p$$

and the result follows from the fact that the composite map

$$\mathbb{F}_p \xrightarrow{1 \otimes \text{id}_{\mathbb{F}_p}} \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\text{mult}} \mathbb{F}_p$$

of animated rings is identity.  $\square$

**Proposition 6.25.** *The map  $\text{THH}(\mathbb{F}_p)^{tC_p} \rightarrow \mathbb{Z}^{tC_p}$  of  $\mathbb{E}_\infty$ - $\text{THH}(\mathbb{Z})^{tC_p}$ -algebras in Construction 6.16 is an equivalence.*

**Proof.** Since both the source and the target is  $p$ -complete by [NS18, Lem I.2.9], it suffices to check that this map is an equivalence after modulo  $p$ . Now the result follows from Lemmas 6.21 and 6.24.  $\square$

**Corollary 6.26.** *Let  $M$  be a  $\text{THH}(\mathbb{Z})^h$ -module in  $\text{Sp}^{g < \mathbb{T}}$ . Then there exists a lax symmetric monoidal (in  $M$ ) equivalence*

$$\left( (M \otimes_{\text{THH}(\mathbb{Z})^h}^{\mathbb{L}} \text{THH}(\mathbb{F}_p)^h)^{\theta_{\text{THH}(\mathbb{F}_p)^{\mathbb{T}}}} \right)_p^\wedge \rightarrow \left( (M \otimes_{\text{THH}(\mathbb{Z})^h}^{\mathbb{L}} \mathbb{Z})^{\theta_{\mathbb{Z}} \mathbb{T}} \right)_p^\wedge$$

of  $\text{TP}(\mathbb{Z})$ -modules.

**Proof.** By Corollary 5.26, it suffices to produce a symmetric monoidal equivalence

$$(M \otimes_{\text{THH}(\mathbb{Z})^h}^{\mathbb{L}} \text{THH}(\mathbb{F}_p)^h)^{\Phi C_p} \longrightarrow (M \otimes_{\text{THH}(\mathbb{Z})^h}^{\mathbb{L}} \mathbb{Z})^{\Phi C_p}$$

in  $\text{Mod}_{\text{THH}(\mathbb{Z})^{tC_p}}(\text{Sp}^{B(\mathbb{T}/C_p)})$ , which is an immediate consequence of Proposition 6.25.  $\square$

**Remark 6.27.** When  $p$  is an odd prime, [DR25, Prop 5.2.1] shows that the map  $\mathbb{Z} \rightarrow \text{THH}(\mathbb{F}_p)$  induces an equivalence  $\mathbb{Z}^{tC_p} \rightarrow \text{THH}(\mathbb{F}_p)^{tC_p}$  of  $\mathbb{E}_\infty$ - $\text{THH}(\mathbb{Z})^{tC_p}$ -algebras, which implies Corollary 6.26 (although a priori, the map there might be different from the one here), and the map even has a  $\mathbb{Z}^{t\mathbb{T}}$ -module structure.

**Corollary 6.28.** *Let  $\mathcal{C}$  be a dualizable presentable stable  $\mathbb{Z}$ -linear  $\infty$ -category. Then there exists a lax symmetric monoidal (in  $\mathcal{C}$ ) equivalence*

$$\text{TP}^{\text{poly}/\mathbb{F}_p}(\mathcal{C} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p)_p^\wedge \longrightarrow \text{HP}^{\text{poly}}(\mathcal{C}/\mathbb{Z})_p^\wedge$$

of  $\text{TP}(\mathbb{Z})$ -modules.

**Proof.** Apply Corollary 6.26 to  $M = \text{THH}(\mathcal{C}) \otimes_{\text{THH}(\mathbb{Z})}^{\mathbb{L}} \text{THH}(\mathbb{Z})^h$ .  $\square$

## 7 Comparison to THH

Recall that the *Cartier isomorphism* identifies algebraic de Rham cohomology groups of smooth  $\mathbb{F}_p$ -algebras with their algebraic differential forms. We give two noncommutative analogues. In this section, we discuss one of them, which compares polynomial coproduct cyclic homology of dualizable presentable stable  $\mathbb{F}_p$ -linear  $\infty$ -categories with their topological Hochschild homology. This comparison was proved in [Kal20, Cor 11.15] for associative  $\mathbb{F}_p$ -algebras, using Goodwillie derivative of (Hochschild-)Witt trace theory. We give two arguments. Although the second argument is much shorter, the first argument gives us more information, which is used in Section 8.

The key to the first argument is the observation that the  $\mathbb{T}/C_p$ -equivariant  $\mathbb{E}_1$ - $\mathbb{Z}^{tC_p}$ -module  $\mathbb{F}_p^{tC_p}$  is co-induced.

**Construction 7.1.** Let  $n \in \mathbb{N}_{>0}$ . There is a map

$$\mathbb{Z}^{\mathbb{T}/C_n} \longrightarrow \mathbb{Z}/n$$

in  $D(\mathbb{Z})^{B\mathbb{T}}$ , where  $\mathbb{Z}/n$  is equipped with the trivial action, i.e.  $\varpi_1^*(\mathbb{Z}/n)$ . Indeed, this map is taken to be represented<sup>7.1</sup> by the following surjective map

$$\begin{array}{ccc} 0 & & -1 \\ \mathbb{Z} & \xrightleftharpoons[n]{0} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}/n & \xrightleftharpoons[0]{0} & 0 \end{array}$$

of mixed complexes concentrated in homological degrees  $[-1, 0]$ . Since it is surjective, it is a fibration in the projective model structure on mixed complexes, with kernel being the mixed complex  $\left( n \mathbb{Z} \xrightleftharpoons[n]{0} \mathbb{Z} \right)$ , which represents  $\mathbb{Z}^{\mathbb{T}} \in D(\mathbb{Z})^{B\mathbb{T}}$ . Consequently, we get a fiber sequence

$$\mathbb{Z}^{\mathbb{T}} \longrightarrow \mathbb{Z}^{\mathbb{T}/C_n} \longrightarrow \mathbb{Z}/n$$

in  $D(\mathbb{Z})^{B\mathbb{T}}$ .

**Remark 7.2.** One can also construct the map and the fiber sequence in Construction 7.1 by the *gold relation*  $a_{\lambda^n} u_{\lambda^m} = (n/m) a_{\lambda^m} u_{\lambda^n}$  for  $(m, n) \in \mathbb{N}_{>0}^2$  with  $m \mid n$  in [HHR17, Lem 3.6] (where  $u_{\lambda^m}$ 's become equivalence after forgetting to the Borel equivariant objects), where  $\lambda^n$  is the complex  $S^1$ -representation  $S^1 \xrightarrow{(-)^n} \mathbb{C}^\times$  viewed as a real representation, and  $a_{\lambda^n}$  is the *Euler class*  $\mathbb{S}^0 \rightarrow \mathbb{S}^{\lambda^n}$ , i.e. the map of Thom spectra of the inclusion  $\{0\} \subseteq \lambda^n$  of representations. Indeed, there is a fiber sequence

$$\Sigma_{\mathbb{T}}^\infty [\mathbb{T}/C_n]_+ \longrightarrow \mathbb{S}^0 \longrightarrow \mathbb{S}^{\lambda^n}$$

of cyclonic spectra (and thus, of  $\mathbb{T}$ -equivariant spectra), cf. [Sul20, Obs 2.32]. Taking  $\mathbb{Z}$ -linear dual, and applying the “octahedral axiom” to the gold relation  $a_{\lambda^n} \sim n a_{\lambda^1}$ , we get the fiber sequence in Construction 7.1.

This allows us to establish results beyond char  $p$ , as observed by Yuri SULYMA in [Sul20, Lem 4.9].

**Corollary 7.3.** *The map  $\mathbb{Z}^{\mathbb{T}/C_n} \rightarrow \mathbb{Z}/n$  in  $D(\mathbb{Z})^{B\mathbb{T}}$  as in Construction 7.1 induces an equivalence*

$$\mathbb{Z}^{tC_n} \otimes_{\mathbb{Z}} \mathbb{Z}^{\mathbb{T}/C_n} \longrightarrow (\mathbb{Z}/n)^{tC_n}$$

in  $\text{Mod}_{\mathbb{Z}^{tC_n}}(D(\mathbb{Z})^{B(\mathbb{T}/C_n)})$ .

**Proof.** Applying  $(-)^{tC_n}$  to the fiber sequence in Construction 7.1, we see that the induced map  $(\mathbb{Z}^{\mathbb{T}/C_n})^{tC_n} \rightarrow (\mathbb{Z}/n)^{tC_n}$  in  $D(\mathbb{Z})^{B(\mathbb{T}/C_n)}$  is an equivalence (which even has an  $\mathbb{E}_1$ -structure). It suffices to establish an equivalence

$$\mathbb{Z}^{tC_n} \otimes_{\mathbb{Z}} \mathbb{Z}^{\mathbb{T}/C_n} \longrightarrow (\mathbb{Z}^{\mathbb{T}/C_n})^{tC_n}$$

in  $\text{Mod}_{\mathbb{Z}^{tC_n}}(D(\mathbb{Z})^{B(\mathbb{T}/C_n)})$ . This is given by equivalences

$$\begin{aligned} \mathbb{Z}^{tC_n} \otimes_{\mathbb{Z}} \mathbb{Z}^{\mathbb{T}/C_n} &\simeq (\mathbb{Z}^{tC_n} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{T}/C_n])[-1] \\ &\simeq (\mathbb{Z}^{tC_n} \otimes (\mathbb{T}/C_n))[-1] \\ &\simeq (\mathbb{Z}[\mathbb{T}/C_n])^{tC_n}[-1] \\ &\simeq (\mathbb{Z}^{\mathbb{T}/C_n})^{tC_n} \end{aligned}$$

in  $\text{Mod}_{\mathbb{Z}^{tC_n}}(D(\mathbb{Z})^{B(\mathbb{T}/C_n)})$ . □

---

7.1. The  $\mathbb{E}_1$ -monoidal equivalence of the monoidal  $\infty$ -category  $D(\mathbb{Z})^{B\mathbb{T}}$  and the monoidal  $\infty$ -category of mixed complexes up to quasi-isomorphism is explained in details in [Lei22, §5.4].

**Warning 7.4.** The map in Corollary 7.3 does not carry any  $\mathbb{E}_1$ -structure when  $n = 2$ . Indeed, on the left hand side, the square-zero class  $e \in \pi_{-1}(\mathbb{Z}^{\mathbb{T}/C_n})$  gives rise to a square-zero class  $e \in \pi_{-1}(\mathbb{Z}^{\mathbb{T}/C_n} \otimes_{\mathbb{Z}} \mathbb{Z}^{tC_n})$ , while the homotopy ring  $\pi_*(\mathbb{F}_2^{tC_2})$  is isomorphic to  $\mathbb{F}_2((s))$  for a generator  $s \in \pi_1(\mathbb{F}_2^{tC_2})$ , which is integral, thus any square-zero elements is necessarily zero.

**Question 5.** Is there any version of  $\mathbb{E}_1$ -enhancement of Corollary 7.3?

Now we compare polynomial periodic cyclic homology of dualizable presentable stable  $\mathbb{F}_p$ -linear  $\infty$ -categories. Slightly more generally, we consider  $\mathrm{THH}(\mathbb{F}_p)$ -modules in cyclonic spectra. By Lemma 2.13 and the symmetric monoidal structure on  $(-)^{\Phi_{C_p}}$ , we have

**Lemma 7.5.** *Let  $k$  be a commutative algebra. Then the composite functor*

$$\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{g < \mathbb{T}}) \xrightarrow{(-) \otimes_{\mathrm{THH}(k)}^{\mathbb{L}} k} \mathrm{Mod}_k(\mathrm{Sp}^{g < \mathbb{T}}) \xrightarrow{((-)^{\theta_k^{\mathbb{T}}})_p^{\wedge}} D(k^{t\mathbb{T}})_p^{\wedge}.$$

*is equivalent to the composite functor*

$$\begin{array}{ccc} \mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{g < \mathbb{T}}) & \xrightarrow{(-)^{\Phi_{C_p}}} & \mathrm{Mod}_{\mathrm{THH}(k)^{\Phi_{C_p}}}(\mathrm{Sp}^{g_p(\mathbb{T}/C_p)}) \\ & \downarrow & \\ & \mathrm{Mod}_{k^{tC_p}}(\mathrm{Sp}^{g_p(\mathbb{T}/C_p)}) & \xrightarrow{(-)^{\eta_{k^{tC_p}(\mathbb{T}/C_p)}}} D(k^{t\mathbb{T}}) \end{array}$$

*where the vertical arrow is the base change along the map  $\mathrm{THH}(k)^{\Phi_{C_p}} \rightarrow k^{tC_p}$  of  $\mathbb{T}/C_p$ - $\mathbb{E}_{\infty}$ -rings.*

**Remark 7.6.** Recall that both maps in the composite map  $\mathbb{Z} \rightarrow \mathbb{Z}_p = \mathrm{TR}(\mathbb{F}_p) \rightarrow \mathrm{THH}(\mathbb{F}_p)$  of  $\mathbb{T}$ - $\mathbb{E}_{\infty}$ -rings become equivalences after taking geometric fixed points  $(-)^{\Phi_{C_p}}$  (cf. [AMR21, Rem 10.9]). It follows that we can identify the  $\mathbb{T}$ - $\mathbb{E}_{\infty}$ -ring  $\mathrm{THH}(\mathbb{F}_p)$  with  $\mathbb{Z}^{\Phi_{C_p}}$ . Moreover, by [HM97], there exists a Bökstedt element  $\sigma \in \pi_2 \mathrm{TF}(\mathbb{F}_p)$  which maps to a Bökstedt element in  $\pi_2 \mathrm{TR}^r(\mathbb{F}_p)$  for every  $r \in \mathbb{N}_{>0}$ , and by [NS18, Lem II.6.1] and computational results about  $\mathrm{THH}(\mathbb{F}_p)$ , we can identify the  $\mathbb{T}$ - $\mathbb{E}_{\infty}$ -ring  $\mathrm{THH}(\mathbb{F}_p)[\sigma^{-1}]$  with the Borel  $\mathbb{T}$ - $\mathbb{E}_{\infty}$ -ring  $\mathbb{Z}^{tC_p}$ . Consequently, for every associative ring  $k$ , the map  $k^{\Phi_{C_p}} \rightarrow k^{tC_p}$  of  $\mathbb{T}$ - $\mathbb{E}_{\infty}$ -rings is simply inverting  $\sigma$ .

**Lemma 7.7.** *The composite functor*

$$\mathrm{Mod}_{\mathrm{THH}(\mathbb{F}_p)^{tC_p}}(\mathrm{Sp}^{g_p(\mathbb{T}/C_p)}) \longrightarrow \mathrm{Mod}_{\mathbb{F}_p^{tC_p}}(\mathrm{Sp}^{g_p(\mathbb{T}/C_p)}) \xrightarrow{(-)^{\eta_{\mathbb{F}_p^{tC_p}(\mathbb{T}/C_p)}}} D(\mathbb{Z}^{t\mathbb{T}})$$

*coincides with the forgetful functor, where the first functor is the base change along the map  $\mathrm{THH}(\mathbb{F}_p)^{tC_p} \rightarrow \mathbb{F}_p^{tC_p}$  of Borel  $\mathbb{E}_{\infty}$ - $p$ -cyclonic spectra.*

Recall that the map  $\mathbb{Z} \rightarrow \mathrm{THH}(\mathbb{F}_p)$  as in Remark 7.6 also becomes an equivalence after taking  $(-)^{tC_p}$  by [NS18, Cor IV.4.13], thus we can replace  $\mathrm{THH}(\mathbb{F}_p)^{tC_p}$  by  $\mathbb{Z}^{tC_p}$  in Lemma 7.7.

**Proof.** Since both functors in question preserve filtered colimits, it suffices to check on compact objects of  $\mathrm{Mod}_{\mathbb{Z}^{tC_p}}(\mathrm{Sp}^{g_p(\mathbb{T}/C_p)})$ . Note that the first base change functor preserves compact objects. The result follows from Corollary 7.3.  $\square$

**Remark 7.8.** The target of the composite functor in Lemma 7.7 has an  $\mathbb{F}_p^{t\mathbb{T}}$ -module structure, and such a structure is yet to explore.

Summarizing the above discussion, we get

**Proposition 7.9.** *The composite functor*

$$\mathrm{Mod}_{\mathrm{THH}(\mathbb{F}_p)}(\mathrm{Sp}^{g < \mathbb{T}}) \xrightarrow{(-) \otimes_{\mathrm{THH}(\mathbb{F}_p)}^{\mathbb{L}} \mathbb{F}_p} \mathrm{Mod}_{\mathbb{F}_p}(\mathrm{Sp}^{g < \mathbb{T}}) \xrightarrow{(-)^{\theta_{\mathbb{F}_p}^{\mathbb{T}}}} D(\mathbb{Z}^{t\mathbb{T}})$$

coincides with the composite functor

$$\mathrm{Mod}_{\mathrm{THH}(\mathbb{F}_p)}(\mathrm{Sp}^{g < \mathbb{T}}) \xrightarrow{(-)^{\Phi_{C_p}}} \mathrm{Mod}_{\mathrm{THH}(\mathbb{F}_p)^{\Phi_{C_p}}}(\mathrm{Sp}^{g_p(\mathbb{T}/C_p)}) \longrightarrow D(\mathbb{Z}^{t\mathbb{T}}),$$

where the second functor is inverting  $\sigma$  to the underlying  $\mathrm{THH}(\mathbb{F}_p)^{\Phi_{C_p}}$ -module.

Applying to the  $\mathrm{THH}(\mathbb{F}_p)$ -module spectrum  $\mathrm{THH}(\mathcal{C})$ , we get:

**Corollary 7.10.** *Let  $\mathcal{C}$  be a dualizable presentable stable  $\mathbb{F}_p$ -linear  $\infty$ -category. Then the polynomial periodic cyclic homology  $\mathrm{HP}^{\mathrm{poly}}(\mathcal{C}/\mathbb{F}_p)$  is equivalent to  $\mathrm{THH}(\mathcal{C})[\sigma^{-1}]$  as  $\mathbb{Z}^{t\mathbb{T}}$ -module spectra.*

**Remark 7.11.** We do not compare the multiplicative structures on polynomial periodic cyclic homology (as a lax symmetric monoidal functor) with topological Hochschild homology in Corollary 7.10.

Now we give a second proof of Lemma 7.7, inspired by the proof of [Mat20, Prop 2.15]. The key is the following lemma.

**Lemma 7.12.** *The composite functor*

$$\mathrm{Mod}_{\mathrm{THH}(\mathbb{F}_p)^{tC_p}}(\mathrm{Sp}^{B(\mathbb{T}/C_p)}) \longrightarrow \mathrm{Mod}_{\mathbb{F}_p^{tC_p}}(\mathrm{Sp}^{B(\mathbb{T}/C_p)}) \xrightarrow{(-)^{h(\mathbb{T}/C_p)}} D(\mathrm{TP}(\mathbb{F}_p))$$

coincides with the forgetful functor.

**Proof.** Since the image of  $u \in \pi_2 \mathrm{TC}^-(\mathbb{F}_p)$  under the canonical map  $\mathrm{TC}^-(\mathbb{F}_p) \rightarrow \mathrm{TP}(\mathbb{F}_p)$  is  $up \in \pi_2 \mathrm{TP}(\mathbb{F}_p)$  as reviewed in Remark 4.10, this composite functor coincides with the composite functor

$$\mathrm{Mod}_{\mathrm{THH}(\mathbb{F}_p)^{tC_p}}(\mathrm{Sp}^{B(\mathbb{T}/C_p)}) \xrightarrow{(-)^{h(\mathbb{T}/C_p)}} D(\mathrm{TP}(\mathbb{F}_p)) \xrightarrow{(-)^{\mathbb{L}_p}} D(\mathrm{TP}(\mathbb{F}_p)),$$

which is subsequently identified with the forgetful functor by the proof of [BMS19, Prop 6.4] (or more precisely, the first displayed formula there).  $\square$

Lemma 7.7 follows from the fact that the composite functor in Lemma 7.12, by virtue of identification with the forgetful functor, preserves filtered colimits, and that the first functor  $\mathrm{Mod}_{\mathrm{THH}(\mathbb{F}_p)^{tC_p}}(\mathrm{Sp}^{g_p(\mathbb{T}/C_p)}) \rightarrow \mathrm{Mod}_{\mathbb{F}_p^{tC_p}}(\mathrm{Sp}^{g_p(\mathbb{T}/C_p)})$  in Lemma 7.7 preserves compact objects.

## 8 Conjugate filtration

As explained in the introduction of Section 7, another noncommutative analogue of the Cartier isomorphism is conjugate filtration, which we will address in this section. As there, our version is constructed out of  $\mathrm{THH}(k)$ -modules for a base commutative ring  $k$ , and by Lemma 7.5, it is crucial to analyze the map  $\mathrm{THH}(k)^{\Phi_{C_p}} \rightarrow k^{tC_p}$ , endowing the target  $k^{tC_p}$  a suitable filtration.

First, we note that, the homotopy  $C_p$ -fixed points  $k^{hC_p}$  of a commutative  $\mathbb{F}_p$ -algebra  $k$  is a direct summand of the  $C_p$ -Tate construction  $k^{tC_p}$ .

**Remark 8.1.** Let  $M$  be an  $\mathbb{F}_p$ -vector space. Then it follows from computations that the composite map

$$M^{hC_p} \longrightarrow M^{tC_p} \longrightarrow \tau_{\leq 0} M^{tC_p}$$

of  $\mathbb{F}_p^{hC_p}$ -module spectra is an equivalence. In particular, let  $k$  be a commutative  $\mathbb{F}_p$ -algebra. Then the composite map

$$k^{hC_p} \longrightarrow k^{tC_p} \longrightarrow \tau_{\leq 0} k^{tC_p}$$

of  $k^{hC_p}$ -module spectra is an equivalence, where the first map has an  $\mathbb{E}_\infty$ -structure, and the second map has a  $k^{tC_p}$ -module structure.

Recall that, in Section 7, a key fact is that the  $\mathbb{T}/C_p$ -equivariant  $\mathbb{Z}^{tC_p}$ -module spectrum  $\mathbb{F}_p^{tC_p}$  is co-induced. It is natural to ask whether the same holds for  $k^{tC_p}$  for any commutative  $\mathbb{F}_p$ -algebra  $k$ . We do not know the answer, but expect it to be false, at least functorially in  $k$  (see Remark 8.6). However, note that the  $\mathbb{E}_\infty$ -ring  $k^{tC_p}$  is 2-periodic, and we show that a “fundamental region” of  $k^{tC_p}$  with respect to its 2-periodicity is  $\mathbb{T}/C_p$ -equivariantly  $k$ -linearly co-induced.

**Notation 8.2.** Let  $n$  be a positive integer. Let  $\mathrm{CoInd}_e^{\mathbb{T}/C_n}$  denote right adjoint  $D(\mathbb{Z}) \rightarrow D(\mathbb{Z})^{B(\mathbb{T}/C_n)}$  to the (symmetric monoidal) forgetful functor  $D(\mathbb{Z})^{B(\mathbb{T}/C_n)} \rightarrow D(\mathbb{Z})$ .

**Construction 8.3.** Let  $M$  be a spectrum. Endow  $M$  with trivial  $\mathbb{T}$ -action, we get a map  $\varpi_p^* M \rightarrow M^{hC_p}$  in  $\mathrm{Fun}(B(\mathbb{T}/C_p), \mathrm{Sp})$ , where  $\varpi_p: \mathbb{T}/C_p \rightarrow *$  is the quotient map, which gives rise to a lax symmetric monoidal lax transformation

$$\varpi_p^*(-) \longrightarrow (-)^{hC_p}$$

between lax symmetric monoidal functors  $\mathrm{Sp} \rightarrow \mathrm{Fun}(B(\mathbb{T}/C_p), \mathrm{Sp})$ . On the other hand, there is a non- $\mathbb{T}/C_p$ -equivariant map  $M^{hC_p} \rightarrow M$  which, by adjunction, gives rise to a map  $M^{hC_p} \rightarrow \mathrm{CoInd}_e^{\mathbb{T}/C_p} M$  in  $\mathrm{Fun}(B(\mathbb{T}/C_p), \mathrm{Sp})$  which is functorial in  $M \in \mathrm{Sp}$ , and this has a lax symmetric monoidal structure. In summary, we have a composite lax symmetric natural transformation

$$\varpi_p^*(-) \longrightarrow (-)^{hC_p} \longrightarrow \mathrm{CoInd}_e^{\mathbb{T}/C_p}(-)$$

of lax symmetric monoidal functors  $\mathrm{Sp} \rightarrow \mathrm{Fun}(B(\mathbb{T}/C_p), \mathrm{Sp})$ .

**Lemma 8.4.** Let  $M$  be an  $\mathbb{F}_p$ -vector space. Then the composite map

$$\tau_{\geq -1} M^{hC_p} \longrightarrow M^{hC_p} \longrightarrow \mathrm{CoInd}_e^{\mathbb{T}/C_p} M$$

of  $\mathbb{T}/C_p$ -equivariant  $\mathbb{F}_p$ -module<sup>8.1</sup> spectra is an equivalence, where the second map is as in Construction 8.3.

**Proof.** Since the maps in question are already constructed, to check that it is an equivalence, we could pick a free  $\mathbb{Z}$ -lift  $\tilde{M}$  of  $M$ , namely, a free abelian group  $\tilde{M} = \mathbb{Z}^{\oplus I}$  with  $\tilde{M}/p \cong M$ . Then taking  $I$ -direct sum<sup>8.2</sup> of the equivalence in Corollary 7.3 and truncating at  $[-1, 0]$ , we get an equivalence as an inverse to the composite map in question.  $\square$

It follows from lax symmetric monoidal structures in Construction 8.3 and Lemma 8.4 that

**Corollary 8.5.** Let  $k$  be a commutative  $\mathbb{F}_p$ -algebra. Then there is a canonical equivalence

$$\tau_{\geq -1} k^{hC_p} \longrightarrow \mathrm{CoInd}_e^{\mathbb{T}/C_p} k$$

of  $\mathbb{T}/C_p$ -equivariant  $k$ -modules.

**Remark 8.6.** Let  $M$  be an  $\mathbb{F}_p$ -vector space. It is unclear whether we can functorially identify  $M^{tC_p}$  with the co-induced  $\mathbb{T}/C_p$ -equivariant  $\mathbb{Z}$ -module spectrum  $\mathrm{CoInd}_e^{\mathbb{T}/C_p}(M^{t\mathbb{T}})$  (although we can do it functorially in free  $\mathbb{Z}$ -lifts  $\tilde{M}$ ), let alone  $\mathbb{F}_p$ -linearly. The multiplicative structure is even more complicated.

Now we describe the filtration on  $k^{hC_p}$  and  $k^{tC_p}$  as promised.

**Construction 8.7.** Let  $k$  be a commutative  $\mathbb{F}_p$ -algebra. Then we consider the odd filtration on  $\mathbb{T}/C_p$ -equivariant  $\tau_{\geq 0}(k^{tC_p})$ -module spectra  $k^{hC_p}$ ,  $k^{tC_p}$ , and the canonical  $\mathbb{T}/C_p$ -equivariant  $\tau_{\geq 0}(k^{tC_p})$ -module map  $k^{hC_p} \rightarrow k^{tC_p}$ , given by the odd parts of the Whitehead filtrations, i.e. for  $i \in \mathbb{Z}$ , we have

$$\begin{aligned} \mathrm{Fil}_{\mathrm{odd}}^i k^{hC_p} &:= \tau_{\geq 2i-1} k^{hC_p}; \\ \mathrm{Fil}_{\mathrm{odd}}^i k^{tC_p} &:= \tau_{\geq 2i-1} k^{tC_p}. \end{aligned}$$

8.1. The  $\mathbb{F}_p$ -module structure comes from lax symmetric monoidal structures in Construction 8.3.

8.2. Tate construction preserves infinite direct sum of uniformly  $t$ -bounded objects.

We note that  $\mathrm{Fil}_{\mathrm{odd}}^{>0} k^{hC_p} = 0$ , and the maps  $\mathrm{gr}_{\mathrm{odd}}^i k^{hC_p} \rightarrow \mathrm{gr}_{\mathrm{odd}}^i k^{tC_p}$  are equivalences for  $i \leq 0$ .

**Remark 8.8.** Note that  $\mathrm{gr}_{\mathrm{odd}}^0 k^{tC_p} = \tau_{\geq -1} k^{hC_p}$  is concentrated in degree  $[-1, 0]$ , by [Lur17, Prop 2.2.1.8], the  $\tau_{\geq 0}(k^{tC_p})$ -module structure on  $\mathrm{gr}_{\mathrm{odd}}^0 k^{tC_p}$  descends canonically to a  $\tau_{[0,1]}(k^{tC_p}) = k \otimes_{\mathbb{Z}} \mathbb{F}_p$ -module structure, where the animated ring  $k \otimes_{\mathbb{Z}} \mathbb{F}_p$  is equipped with trivial  $\mathbb{T}/C_p$ -action.

We now produce the conjugate filtration for  $\mathrm{THH}(k)$ -modules. Before this, we need a base-independence result, which says that the de-completed Tate construction over  $k^{tC_p}$  coincides with that over  $\tau_{\geq 0}(k^{tC_p})$ .

**Lemma 8.9.** *Let  $k$  be a commutative  $\mathbb{F}_p$ -algebra. Then the natural transformation*

$$(-)^{\eta_{\tau_{\geq 0}(k^{tC_p})}(\mathbb{T}/C_p)} \longrightarrow (-)^{\eta_{k^{tC_p}}(\mathbb{T}/C_p)}$$

*of functors  $\mathrm{Mod}_{k^{tC_p}}(\mathrm{Sp}^{g_p \mathbb{T}}) \rightarrow D(k)$ , induced by the map  $\tau_{\geq 0}(k^{tC_p}) \rightarrow k^{tC_p}$  in  $\mathrm{CAlg}(D(k)^{B(\mathbb{T}/C_p)})$ , is an equivalence.*

**Proof.** It suffices to show that, for every compact generator  $M = k^{tC_p} \otimes [\mathbb{T}/C_p] \in \mathrm{Mod}_{k^{tC_p}}(\mathrm{Sp}^{g_p \mathbb{T}})^{\aleph_0}$ , the assembly map

$$M^{\eta_{\tau_{\geq 0}(k^{tC_p})}(\mathbb{T}/C_p)} \longrightarrow M^{h(\mathbb{T}/C_p)}$$

is an equivalence. Recall that the Whitehead tower of  $k^{tC_p} = (\tau_{\geq 0} k^{tC_p})[\sigma^{-1}]$  is a sequential colimit of shifts of  $\tau_{\geq 0}(k^{tC_p})$ , where  $\sigma \in \pi_2(k^{tC_p})$  is a generator. Then the result follows from the fact that  $(-)^{h(\mathbb{T}/C_p)}$  preserves weakly Whitehead towers (cf. [NS18, Lem I.2.6], or the dual of [BMS19, Lem 3.3]).  $\square$

**Construction 8.10. (Conjugate filtration)** Let  $k$  be a commutative  $\mathbb{F}_p$ -algebra, and  $M$  a  $\mathrm{THH}(k)$ -module in  $\mathrm{Sp}^{g < \mathbb{T}}$ . Then by Lemma 7.5 and Construction 8.7, the  $\mathbb{Z}^{t\mathbb{T}}$ -module spectrum

$$(M \otimes_{\mathrm{THH}(k)} k)^{\theta_k \mathbb{T}} \simeq (M^{\Phi C_p} \otimes_{\mathrm{THH}(k)^{\Phi C_p}} \tau_{\geq 0}(k^{tC_p}) \otimes_{\tau_{\geq 0}(k^{tC_p})} k^{tC_p})^{\eta_{\tau_{\geq 0}(k^{tC_p})}(\mathbb{T}/C_p)}$$

admits a filtration  $\mathrm{Fil}_{\mathrm{conj}}^*$  induced by the odd filtration on  $k^{tC_p}$ .

We now analyze the associated graded pieces of the conjugate filtration. It suffices to analyze the zeroth associated piece:

**Remark 8.11.** By 2-periodicity of  $k^{tC_p}$ , all  $\mathrm{gr}_{\mathrm{odd}}^i k^{tC_p}$ 's, as  $k^{\Phi C_p}$ -modules, are isomorphic up to a shift. Thus to study associated graded pieces of odd filtered  $k^{tC_p}$ , it suffices to study  $\mathrm{gr}_{\mathrm{odd}}^0 k^{tC_p} \simeq \mathrm{gr}_{\mathrm{odd}}^i k^{hC_p}$ , thus the same for the conjugate filtration.

Since  $k \otimes_{\mathbb{Z}} \mathbb{F}_p = (\tau_{\geq 0} k^{tC_p})/\sigma$  is a perfect  $\tau_{\geq 0} k^{tC_p}$ -module in  $\mathrm{Sp}^{B(\mathbb{T}/C_p)}$ , we have

**Lemma 8.12.** *Let  $k$  be a commutative  $\mathbb{F}_p$ -algebra. Then the natural transformation*

$$(-)^{\eta_{\tau_{\geq 0}(k^{tC_p})}(\mathbb{T}/C_p)} \longrightarrow (-)^{\eta_{k \otimes_{\mathbb{Z}} \mathbb{F}_p}(\mathbb{T}/C_p)}$$

*of functors  $\mathrm{Mod}_{k \otimes_{\mathbb{Z}} \mathbb{F}_p}(\mathrm{Sp}^{g_p \mathbb{T}}) \rightarrow D(k)$ , induced by the map  $\tau_{\geq 0}(k^{tC_p}) \rightarrow k \otimes_{\mathbb{Z}} \mathbb{F}_p$  in  $\mathrm{CAlg}(D(k)^{B(\mathbb{T}/C_p)})$ , is an equivalence (we tacitly used Remark 2.4).*

Similarly, by perfectness of  $k$ -module  $k \otimes_{\mathbb{Z}} \mathbb{F}_p$ , we have

**Lemma 8.13.** *Let  $k$  be a commutative  $\mathbb{F}_p$ -algebra. Then the natural transformation*

$$(-)^{\eta_k(\mathbb{T}/C_p)} \longrightarrow (-)^{\eta_{k \otimes_{\mathbb{Z}} \mathbb{F}_p}(\mathbb{T}/C_p)}$$

*of functors  $\mathrm{Mod}_{k \otimes_{\mathbb{Z}} \mathbb{F}_p}(\mathrm{Sp}^{g_p \mathbb{T}}) \rightarrow D(k)$ , induced by the map  $k \rightarrow k \otimes_{\mathbb{Z}} \mathbb{F}_p$  in  $\mathrm{CAlg}(D(k)^{B(\mathbb{T}/C_p)})$ , is an equivalence (we tacitly used Remark 2.4).*

It follows from Remark 8.8 and Lemmas 8.12 and 8.13 that



**Lemma 8.14.** *Let  $k$  be a commutative  $\mathbb{F}_p$ -algebra, and  $M$  a  $\mathrm{THH}(k)$ -module in  $\mathrm{Sp}^{g^{<\mathbb{T}}}$ . Then the 0-th associated graded piece  $\mathrm{gr}_{\mathrm{conj}}^0(M \otimes_{\mathrm{THH}(k)} \underline{k})^{\theta_k \mathbb{T}}$  of the conjugate filtration is equivalent to*

$$((M^{\Phi_{C_p}} \otimes_{\mathrm{THH}(k)^{\Phi_{C_p}}} (k \otimes_{\mathbb{Z}} \mathbb{F}_p)) \otimes_{k \otimes_{\mathbb{Z}} \mathbb{F}_p} \mathrm{gr}_{\mathrm{odd}}^0 k^{tC_p})^{\eta_k(\mathbb{T}/C_p)}$$

in  $D(k)$  (again, we tacitly used Remark 2.4).

Now we embark to understand the composite map  $\mathrm{THH}(k) \xrightarrow{\simeq} \mathrm{THH}(k)^{\Phi_{C_p}} \rightarrow k \otimes_{\mathbb{Z}} \mathbb{F}_p$  of  $\mathbb{T}$ - $\mathbb{E}_{\infty}$ -rings<sup>8.3</sup>. In fact, this could be understood for any commutative ring, not necessarily over  $\mathbb{F}_p$ .

**Remark 8.15.** Let  $k$  be an animated ring. By construction (cf. [NS18, §IV.2]), we have a commutative diagram

$$\begin{array}{ccccc} k & \longrightarrow & \mathrm{THH}(k) & & \\ \downarrow \simeq & & \downarrow \simeq & & \\ (N_e^{C_p} k)^{\Phi_{C_p}} & \longrightarrow & \mathrm{THH}(k)^{\Phi_{C_p}} & \longrightarrow & \underline{k}^{\Phi_{C_p}} \\ \downarrow & & \downarrow & & \downarrow \\ (k^{\otimes_{\mathrm{Sp}}} tC_p) & \longrightarrow & \mathrm{THH}(k)^{tC_p} & \longrightarrow & k^{tC_p} \end{array}$$

where the leftmost row are maps of  $\mathbb{E}_{\infty}$ -rings while the rest are  $\mathbb{T}$ - $\mathbb{E}_{\infty}$ -rings. The composite map  $k \rightarrow k^{tC_p}$  from the top left to the bottom right is the Tate-valued Frobenius.

**Remark 8.16.** Let  $k$  be an animated ring. Note that we have a commutative diagram

$$\begin{array}{ccc} k & \longrightarrow & \underline{k}^{\Phi_{C_p}} \\ \downarrow & & \downarrow \\ \underline{k} & \xrightarrow{\tilde{\varphi}_k} & k \otimes_{\mathbb{Z}} \mathbb{F}_p \end{array}$$

in  $\mathrm{CAlg}(D(\mathbb{Z}))$ , where the top arrow is found in Remark 8.15, the bottom arrow is induced by the “extended” Frobenius map  $\tilde{\varphi}_k: k \rightarrow k \otimes_{\mathbb{Z}} \mathbb{F}_p \rightarrow k \otimes_{\mathbb{Z}} \mathbb{F}_p$  of animated rings, and except the top left  $k$ , other terms are  $\mathbb{T}$ - $\mathbb{E}_{\infty}$ -rings, and the arrows between  $\mathbb{T}$ - $\mathbb{E}_{\infty}$ -rings have  $\mathbb{T}$ - $\mathbb{E}_{\infty}$ -structures. By the universal property of  $\mathrm{THH}$  in [ABG+18], we get a commutative diagram

$$\begin{array}{ccc} \mathrm{THH}(k) & \longrightarrow & \underline{k}^{\Phi_{C_p}} \\ \downarrow & & \downarrow \\ \underline{k} & \xrightarrow{\tilde{\varphi}_k} & k \otimes_{\mathbb{Z}} \mathbb{F}_p \end{array}$$

of  $\mathbb{T}$ - $\mathbb{E}_{\infty}$ -rings<sup>8.4</sup>.

So far, we have analyze the part  $M^{\Phi_{C_p}} \otimes_{\mathrm{THH}(k)^{\Phi_{C_p}}} (k \otimes_{\mathbb{Z}} \mathbb{F}_p)$  of the expression in Lemma 8.14. Now we analyze the part  $((-) \otimes_{k \otimes_{\mathbb{Z}} \mathbb{F}_p} \mathrm{gr}_{\mathrm{odd}}^0 k^{tC_p})^{\eta_k(\mathbb{T}/C_p)}$  for commutative  $\mathbb{F}_p$ -algebras  $k$ . It follows from Corollary 8.5 that this simplifies to  $(-) \otimes_{k \otimes_{\mathbb{Z}} \mathbb{F}_p} k$ , where the  $k \otimes_{\mathbb{Z}} \mathbb{F}_p$ -module structure on  $k$  is simply given by the multiplication map  $k \otimes_{\mathbb{Z}} \mathbb{F}_p \rightarrow k$ . Recall that, for animated  $\mathbb{F}_p$ -algebras  $k$ , the composite map

$$k \xrightarrow{\tilde{\varphi}_k} k \otimes_{\mathbb{Z}} \mathbb{F}_p \longrightarrow k$$

coincides with the usual Frobenius map  $\varphi_k: k \rightarrow k$ . The above discussion implies that

**Proposition 8.17.** *Let  $k$  be a commutative  $\mathbb{F}_p$ -algebra, and  $M$  a  $\mathrm{THH}(k)$ -module in  $\mathrm{Sp}^{g^{<\mathbb{T}}}$ . Then the 0-th associated graded piece  $\mathrm{gr}_{\mathrm{conj}}^0(M \otimes_{\mathrm{THH}(k)} \underline{k})^{\theta_k \mathbb{T}}$  of the conjugate filtration is equivalent to the Frobenius twist*

$$(M^{\Phi_{C_p}} \otimes_{\mathrm{THH}(k)} k) \otimes_{k, \varphi_k} k$$

<sup>8.3</sup> For our purposes, thanks to Remark 2.4, it suffices to understand the underlying  $\mathbb{T}$ -equivariant  $\mathbb{E}_{\infty}$ -ring. However, here we deduce slightly stronger results.

<sup>8.4</sup> Those who are not familiar with  $\mathbb{T}$ - $\mathbb{E}_{\infty}$ -rings can simply replace them by  $\mathbb{T}$ -equivariant  $\mathbb{E}_{\infty}$ -rings, and the  $\mathrm{THH}$  also has an analogous universal property by McClure–Schwänzl–Vogt, cf. [NS18, Prop IV.2.2].

in  $D(k)$ , where the  $\mathrm{THH}(k)$ -module structure on  $M^{\Phi C_p}$  is induced by the equivalence  $\mathrm{THH}(k) \xrightarrow{\simeq} \mathrm{THH}(k)^{\Phi C_p}$ .

**Corollary 8.18.** *Let  $k$  be a commutative  $\mathbb{F}_p$ -algebra, and  $\mathcal{C}$  a dualizable presentable stable  $\mathbb{F}_p$ -linear  $\infty$ -category. Then the 0-th associated graded piece  $\mathrm{gr}_{\mathrm{conj}}^0 \mathrm{HP}^{\mathrm{poly}}(\mathcal{C}/k)$  is equivalent to  $\mathrm{HH}(\mathcal{C}/k) \otimes_{k, \varphi_k}^{\mathbb{L}} k$  in  $D(k)$ .*

Finally, we show that the conjugate filtration is complete. It suffices to establish the following connectivity result.

**Lemma 8.19.** *Let  $k$  be a commutative  $\mathbb{F}_p$ -algebra, and  $M$  a  $\mathrm{THH}(k)^{\Phi C_p}$ -module in  $\mathrm{Sp}^{g < (\mathbb{T}/C_p)}$ . Suppose that  $M$  is connective. Then the spectrum*

$$(M \otimes_{\mathrm{THH}(k)^{\Phi C_p}}^{\mathbb{L}} \tau_{\geq -1} k^{tC_p})^{\eta_{\tau \geq 0}(k^{tC_p})^{(\mathbb{T}/C_p)}}$$

*is connective.*

Note that  $k^{tC_p}$  is co-induced in  $D(\mathbb{Z})$ . Although it might not be compatibility with  $\mathrm{THH}(k)^{\Phi C_p}$ -module structure, this is already enough for us to prove the connectivity.

**Proof.** By the bar resolution of the relative tensor product, and the fact that the de-completed Tate construction preserves small colimits, it suffices to show that, for every  $n \in \mathbb{N}$ , the spectrum

$$(M \otimes_{\mathbb{Z}}^{\mathbb{L}} (\mathrm{THH}(k)^{\Phi C_p})^{\otimes_{\mathbb{Z}} n} \otimes_{\mathbb{Z}}^{\mathbb{L}} \tau_{\geq -1} k^{tC_p})^{\eta_{\tau \geq 0}(k^{tC_p})^{(\mathbb{T}/C_p)}}$$

is connective. We write  $k$  as a direct sum  $\mathbb{F}_p^{\oplus I}$ , and applying Corollary 7.3, we see that this spectrum is equivalent to the direct sum of  $I$ 's copies of

$$M \otimes_{\mathbb{Z}}^{\mathbb{L}} (\mathrm{THH}(k)^{\Phi C_p})^{\otimes_{\mathbb{Z}} n} \otimes_{\mathbb{Z}}^{\mathbb{L}} \tau_{\geq 0} \mathbb{Z}^{tC_p}$$

which is connective.  $\square$

By 2-periodicity of  $k^{tC_p}$ , and the fact that  $(-)^{\Phi C_p}$  preserves connectivity, we deduce from Lemma 8.19 that

**Corollary 8.20.** *Let  $k$  be a commutative  $\mathbb{F}_p$ -algebra, and  $M$  a connective  $\mathrm{THH}(k)$ -module in  $\mathrm{Sp}^{g < \mathbb{T}}$ . Then the conjugate filtration on  $(M \otimes_{\mathrm{THH}(k)}^{\mathbb{L}} k)^{\eta_{\mathbb{Z}} \mathbb{T}}$  is complete.*

**Corollary 8.21.** *Let  $k$  be a commutative  $\mathbb{F}_p$ -algebra, and  $\mathcal{C}$  a dualizable presentable stable  $\mathbb{F}_p$ -linear  $\infty$ -category. Suppose that its topological Hochschild homology  $\mathrm{THH}(\mathcal{C})$  is bounded below<sup>8.5</sup>. Then the conjugate filtration on its polynomial periodic cyclic homology  $\mathrm{HP}^{\mathrm{poly}}(\mathcal{C}/k)$  is complete.*

## Appendix A Tate cohomology complex

We briefly show that our de-completed Tate cohomology on Lazard-semi-flat chain complexes of cohomological Mackey functors can be computed by *Tate cohomology complex* in [Kal15, §6.2], or [PVV18, §2.1], thus it is a homotopy invariant version of the latter.

Let  $k$  be a commutative ring. Recall that a *finitely generated permutation  $G$ -module* is a (left)  $k[G]$ -module of the form  $k[X]$  for some *finite*  $G$ -set  $X$ . We denote by  $\mathrm{Perm}_G(k) \subseteq \mathrm{LMod}_{k[G]}$  the full subcategory spanned by finitely generated permutation  $G$ -modules. We refer to [BG21] for comparison of different characterizations of cohomological Mackey functors, and [BCN21, Ex 2.5] for the equivalence  $D(\mathrm{Mack}_G^{\mathrm{coh}}(k)) = \mathrm{Mod}_{\mathbb{k}}(\mathrm{Sp}^{gG})$  between derived cohomological  $G$ -Mackey functors and  $\mathbb{k}$ -modules in genuine  $G$ -spectra.

Now we start with a flatness of cohomological Mackey functors.

<sup>8.5</sup> This is the case when  $\mathcal{C} = D(R)$  for a  $(-1)$ -connective  $\mathbb{E}_1$ - $k$ -algebra  $R$ , or  $\mathcal{C} = D(X)$  for a quasicompact quasiseperated  $k$ -scheme  $X$ .

**Definition A.1.** Let  $k$  be a commutative ring, and  $G$  a finite group.

- We say that a cohomological Mackey functor is Lazard-flat if it is a filtered colimit of finitely generated permutation modules.
- We say that an (unbounded) chain complex of cohomological Mackey functors is Lazard-semi-flat if it is a filtered colimit of bounded chain complexes of finitely generated permutation modules. Compare with [CH15, Thm 1.1].

**Remark A.2.** Let  $k$  be a commutative ring, and  $G$  a finite group. Then permutation  $k[G]$ -modules are compact in the category  $\text{Mack}_G^{\text{coh}}(k)$  of cohomological  $G$ -Mackey functors. Consequently, we have a canonical fully faithful functor

$$\text{Ind}(\text{Perm}_G) \hookrightarrow \text{Mack}_G^{\text{coh}}(k).$$

In particular, we can identify Lazard-flat (derived) cohomological Mackey functors as objects of  $\text{Ind}(\text{Perm}_G(k))$ .

Similarly, recall that a chain complex in an additive category is compact if and only if it is bounded and degreewise compact [CH15, Thm 4.5]. The preceding argument shows that we have a canonical fully faithful functor

$$\text{Ind}(\text{Ch}^b(\text{Perm}_G(k))) \hookrightarrow \text{Ch}(\text{Mack}_G^{\text{coh}}(k))$$

and thus we can identify Lazard-semi-flat chain complexes of cohomological Mackey functors as objects of  $\text{Ind}(\text{Ch}^b(\text{Perm}_G(k)))$ .

Lazard-flat cohomological Mackey functors (resp. Lazard-semi-flat chain complexes of cohomological Mackey functors) are in fact  $k[G]$ -modules (resp. chain complexes of  $k[G]$ -modules):

**Remark A.3.** Let  $k$  be a commutative ring, and  $G$  a finite group. Then finitely generated permutation  $k[G]$ -modules are compact in the category  $\text{LMod}_{k[G]}$  of left  $G$ -modules, thus the fully faithful functor

$$\text{Ind}(\text{Perm}_G(k)) \hookrightarrow \text{Mack}_G^{\text{coh}}(k)$$

in Remark A.2 factors through the inclusion  $\text{LMod}_{k[G]} \hookrightarrow \text{Mack}_G^{\text{coh}}(k)$ , and the fully faithful functor

$$\text{Ind}(\text{Ch}^b(\text{Perm}_G(k))) \hookrightarrow \text{Ch}(\text{Mack}_G^{\text{coh}}(k))$$

factors through the inclusion  $\text{Ch}(\text{LMod}_{k[G]}) \hookrightarrow \text{Ch}(\text{Mack}_G^{\text{coh}}(k))$ .

Now we review the Tate cohomology complex. Let  $k$  be a commutative ring, and  $G$  a finite group. A *complete resolution* [PVV18, §2.1] of the left  $k[G]$ -module  $k$  is an acyclic complex  $P_* \in \text{Ch}(\text{LMod}_{k[G]}^{\text{free}})$  of free left  $k[G]$ -modules along with an isomorphism  $\varepsilon: \mathbb{Z} \rightarrow \ker(d: P_0 \rightarrow P_{-1})$  of left  $k[G]$ -modules.

**Definition A.4.** Let  $k$  be a commutative ring,  $G$  a finite group, and  $P_*$  a complete resolution of  $k$ . The functor of the Tate cohomology complex is defined to be the functor

$$\begin{aligned} \text{Ch}(\text{Mack}_G^{\text{coh}}(k)) &\longrightarrow \text{Ch}(k), \\ M_* &\longmapsto (\text{Tot}^{\oplus}(M_* \otimes_k P_*))^G. \end{aligned}$$

On Lazard-semi-flat chain complexes of cohomological Mackey functors, the Tate cohomology complex represents the de-completed Tate cohomology. More precisely, we have

**Proposition A.5.** Let  $k$  be a commutative ring, and  $G$  a finite group. Then we have a commutative diagram

$$\begin{array}{ccc} \text{Ind}(\text{Ch}^b(\text{Perm}_G(k))) & \longrightarrow & \text{Ch}(k) \\ \downarrow & & \downarrow \\ D(\text{Mack}_G^{\text{coh}}(k)) = \text{Mod}_{\mathbb{k}}(\text{Sp}^{gG}) & \xrightarrow{(-)^{\theta_{\mathbb{k}}^G}} & D(k) \end{array}$$

of additive  $\infty$ -categories, where the top horizontal arrow is the composite of the inclusion in Remark A.2 and the Tate cohomology complex, and the vertical arrows are localizations at quasi-isomorphisms.

**Sketch of proof.** Since all functors in question preserve filtered colimits, we can replace the top left term  $\mathrm{Ind}(\mathrm{Ch}^b(\mathrm{Perm}_G(k)))$  by  $\mathrm{Ch}^b(\mathrm{Perm}_G(k))$ . By stability, we could further restrict to the full subcategory  $\mathrm{Ch}_{\geq 0}^b(\mathrm{Perm}_G(k))$  of bounded chain complexes concentrated in non-negative degrees. This is the same as freely adjoining finitary-geometric-realizations to  $\mathrm{Perm}_G(k)$ . All functors in question preserve finitary geometric realizations, thus we can replace the top left term simply by  $\mathrm{Perm}_G(k)$ .

In this case, the left vertical arrow is fully faithful, with essential image being compact in  $\mathrm{Mod}_{\mathbb{k}}(\mathrm{Sp}^{gG})$ . Therefore we may replace  $(-)^{\theta_{\mathbb{k}}^G}$  by  $(-)^{tG}$ . It remains to show that, for every finitely generated permutation  $G$ -module  $M$ , the complex  $(M \otimes_k P_*)^G$  represents  $M^{tG}$ , which follows from definition.  $\square$

## Bibliography

- [ABG+18] Vigleik Angeltveit, Andrew J. Blumberg, Teena Gerhardt, Michael A. Hill, Tyler Lawson, and Michael A. Mandell. Topological cyclic homology via the norm. *Doc. Math.*, 23:2101–2163, 2018.
- [ACH24] Gabriel Angelini-Knoll, Dominic Leon Culver, and Eva Höning. Topological Hochschild homology of truncated Brown-Peterson spectra, I. *Algebr. Geom. Topol.*, 24(5):2509–2536, 2024.
- [AMMN22] Benjamin Antieau, Akhil Mathew, Matthew Morrow, and Thomas Nikolaus. On the Beilinson fiber square. *Duke Math. J.*, 171(18):3707–3806, 2022.
- [AMR17a] David Ayala, Aaron Mazel-Gee, and Nick Rozenblyum. A naive approach to genuine  $G$ -spectra and cyclotomic spectra. *ArXiv e-prints*, oct 2017.
- [AMR17b] David Ayala, Aaron Mazel-Gee, and Nick Rozenblyum. The geometry of the cyclotomic trace. *ArXiv e-prints*, oct 2017.
- [AMR21] David Ayala, Aaron Mazel-Gee, and Nick Rozenblyum. Derived Mackey functors and  $C_{p^n}$ -equivariant cohomology. *ArXiv e-prints*, may 2021.
- [Ant19] Benjamin Antieau. Periodic cyclic homology and derived de Rham cohomology. *Ann. K-Theory*, 4(3):505–519, 2019.
- [Bal24] Konrad Bals. Periodic cyclic homology over  $\mathbb{Q}$ . *Ann. K-Theory*, 9(1):119–142, 2024.
- [BCN21] Lukas Brantner, Ricardo Campos, and Joost Nuiten. PD Operads and Explicit Partition Lie Algebras. *ArXiv e-prints*, apr 2021.
- [Bei12] A. Beilinson.  $p$ -adic periods and derived de Rham cohomology. *J. Amer. Math. Soc.*, 25(3):715–738, 2012.
- [BG16] Clark Barwick and Saul Glasman. Cyclonic spectra, cyclotomic spectra, and a conjecture of Kaledin. *ArXiv e-prints*, feb 2016.
- [BG21] Paul Balmer and Martin Gallauer. Permutation modules, Mackey functors, and Artin motives. *ArXiv e-prints*, jul 2021.
- [Bha12] B. Bhatt.  $p$ -adic derived de Rham cohomology. *ArXiv e-prints*, apr 2012.
- [BHM22] Andrew J. Blumberg, Michael A. Hill, and Michael A. Mandell. Norms for compact Lie groups in equivariant stable homotopy theory. *ArXiv e-prints*, dec 2022.
- [BL22] Bhargav Bhatt and Jacob Lurie. Absolute prismatic cohomology. *ArXiv e-prints*, jan 2022.
- [BM94] M. Bökstedt and I. Madsen. Topological cyclic homology of the integers. Number 226, pages 7–8. 1994. *K-theory (Strasbourg, 1992)*.
- [BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral  $p$ -adic Hodge theory. *Publications Mathématiques. Institut de Hautes Études Scientifiques*, 129:199–310, 2019.
- [Bor11] James Borger. The basic geometry of Witt vectors, I: The affine case. *Algebra Number Theory*, 5(2):231–285, 2011.
- [BSW17] Serge Bouc, Radu Stancu, and Peter Webb. On the projective dimensions of Mackey functors. *Algebr. Represent. Theory*, 20(6):1467–1481, 2017.
- [BvdB03] A. Bondal and M. van den Bergh. Generators and representability of functors in commutative and noncommutative geometry. *Mosc. Math. J.*, 3(1):1–36, 2003.
- [CH15] Lars Winther Christensen and Henrik Holm. The direct limit closure of perfect complexes. *J. Pure Appl. Algebra*, 219(3):449–463, 2015.
- [Dev23] Sanath K Devalapurkar. Topological Hochschild homology, truncated Brown-Peterson spectra, and a topological Sen operator. *ArXiv e-prints*, mar 2023.
- [DR25] Sanath K. Devalapurkar and Arpon Raksit.  $\mathrm{THH}(\mathbb{Z})$  and the image of  $J$ . *ArXiv e-prints*, may 2025.
- [Ef24] Alexander I. Efimov.  $K$ -theory and localizing invariants of large categories. *ArXiv e-prints*, page 0, may 2024.

- [Gro66] A. Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.
- [Hes96] Lars Hesselholt. On the  $p$ -typical curves in Quillen’s  $K$ -theory. *Acta Math.*, 177(1):1–53, 1996.
- [HHR17] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. The slice spectral sequence for the  $C_4$  analog of real  $K$ -theory. *Forum Mathematicum*, 29(2):383–447, 2017.
- [HLL20] Lars Hesselholt, Michael Larsen, and Ayelet Lindenstrauss. On the  $K$ -theory of division algebras over local fields. *Invent. Math.*, 219(1):281–329, 2020.
- [HM97] Lars Hesselholt and Ib Madsen. On the  $K$ -theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.
- [HN20] Lars Hesselholt and Thomas Nikolaus. Algebraic  $K$ -theory of planar cuspidal curves. In *K-theory in algebra, analysis and topology*, volume 749 of *Contemp. Math.*, pages 139–148. Amer. Math. Soc., 2020.
- [Hoe20] J. van der Hoeven. *The Jolly Writer. Your Guide to GNU TeXmacs*. Scypress, 2020.
- [HRW22] Jeremy Hahn, Arpon Raksit, and Dylan Wilson. A motivic filtration on the topological cyclic homology of commutative ring spectra. *ArXiv e-prints*, jun 2022.
- [HW22] Jeremy Hahn and Dylan Wilson. Redshift and multiplication for truncated Brown-Peterson spectra. *Ann. of Math. (2)*, 196(3):1277–1351, 2022.
- [Ill72] Luc Illusie. *Complexe cotangent et déformations. II*. Lecture Notes in Mathematics, Vol. 283. Springer-Verlag, Berlin-New York, 1972.
- [Kal15] D. Kaledin. Bokstein homomorphism as a universal object. *ArXiv e-prints*, oct 2015.
- [Kal17] D. Kaledin. Spectral sequences for cyclic homology. In *Algebra, geometry, and physics in the 21st century*, volume 324 of *Progr. Math.*, pages 99–129. Birkhäuser/Springer, Cham, 2017.
- [Kal18] D. Kaledin. Co-periodic cyclic homology. *Adv. Math.*, 334:81–150, 2018.
- [Kal20] D. Kaledin. Trace theories, Bokstedt periodicity and Bott periodicity. *ArXiv e-prints*, apr 2020.
- [KMN23] Achim Krause, Jonas McCandless, and Thomas Nikolaus. Polygonic spectra and TR with coefficients. *ArXiv e-prints*, feb 2023.
- [Kon08] Maxim Kontsevich. XI Solomon Lefschetz Memorial Lecture series: Hodge structures in non-commutative geometry. In *Non-commutative geometry in mathematics and physics*, volume 462 of *Contemp. Math.*, pages 1–21. Amer. Math. Soc., Providence, RI, 2008. Notes by Ernesto Lupercio.
- [Lei22] Malte Sander Leip. *On the Hochschild homology of hypersurfaces as a mixed complex, volume 1*. PhD thesis, University of Copenhagen, 2022. <https://noter.math.ku.dk/phd22msl-vol1.pdf>.
- [Lod98] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, Second edition, 1998. Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.
- [Lur11] Jacob Lurie. Derived algebraic geometry XIII: rational and  $p$ -adic homotopy theory. <https://www.math.ias.edu/~lurie/papers/DAG-XIII.pdf>, dec 2011.
- [Lur17] Jacob Lurie. Higher algebra. <https://www.math.ias.edu/~lurie/papers/HA.pdf>, sep 2017.
- [Lur18] Jacob Lurie. Spectral algebraic geometry. <https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf>, feb 2018.
- [Mag24] Kirill Magidson. Divided Powers and Derived De Rham Cohomology. *ArXiv e-prints*, may 2024.
- [Maj96] Javier Majadas. Derived de Rham complex and cyclic homology. *Math. Scand.*, 79(2):176–188, 1996.
- [Man24] Deven Manam. On the Drinfeld formal group. *ArXiv e-prints*, mar 2024.
- [Mao24a] Zhouhang Mao. Noncommutative relative de Rham–Witt complex via the norm. *ArXiv e-prints*, oct 2024.
- [Mao24b] Zhouhang Mao. Prismatic logarithm and prismatic Hochschild homology via the norm. *ArXiv e-prints*, sep 2024.
- [Mat20] Akhil Mathew. Kaledin’s degeneration theorem and topological Hochschild homology. *Geom. Topol.*, 24(6):2675–2708, 2020.
- [MNN19] Akhil Mathew, Niko Naumann, and Justin Noel. Derived induction and restriction theory. *Geom. Topol.*, 23(2):541–636, 2019.
- [NS18] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *Acta Mathematica*, 221(2):203–409, 2018.
- [Orl16] Dmitri Orlov. Smooth and proper noncommutative schemes and gluing of DG categories. *Adv. Math.*, 302:59–105, 2016.
- [PV19] Alexander Petrov and Vadim Vologodsky. On the periodic topological cyclic homology of DG categories in characteristic  $p$ . *ArXiv e-prints*, dec 2019.
- [PVV18] Alexander Petrov, Dmitry Vaintrob, and Vadim Vologodsky. The Gauss-Manin connection on the periodic cyclic homology. *Selecta Math. (N.S.)*, 24(1):531–561, 2018.
- [Rak20] Arpon Raksit. Hochschild homology and the derived de Rham complex revisited. *ArXiv e-prints*, jul 2020.
- [Rig25] Noah Riggenbach.  $K$ -theory of truncated polynomials. *Math. Z.*, 310(3):0, 2025.
- [Sul20] Yuri J. F. Sulyma. A slice refinement of Bökstedt periodicity. *ArXiv e-prints*, jul 2020.
- [TV11] Bertrand Toën and Gabriele Vezzosi. Algèbres simpliciales  $S^1$ -équivariantes, théorie de de Rham et théorèmes HKR multiplicatifs. *Compos. Math.*, 147(6):1979–2000, 2011.